Notes on Equilibria in Symmetric Games

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Abstract

In a symmetric game, every player is identical with respect to the game rules. We show that a symmetric 2-strategy game must have a pure-strategy Nash equilibrium. We also discuss Nash's original paper and its generalized notion of symmetry in games. As a special case of Nash's theorem, any finite symmetric game has a symmetric Nash equilibrium. Furthermore, symmetric infinite games with compact, convex strategy spaces and continuous, quasiconcave utility functions have symmetric pure-strategy Nash equilibria. Finally, we discuss how to exploit symmetry for more efficient methods of finding Nash equilibria.

1. Introduction

A game in normal form is *symmetric* if all agents have the same strategy set, and the payoff to playing a given strategy depends only on the strategies being played, not on who plays them. (We provide a formal definition in the next section.) Many well-known games are symmetric, for example the ubiquitous Prisoners' Dilemma, as well as standard game-theoretic auction models. Symmetric games may naturally arise from models of automated-agent interactions, since in these environments the agents may possess identical circumstances, capabilities, and perspectives, by design. Even when actual circumstances, etc., may differ, it is often accurate to model these as drawn from identical probability distributions. Designers often impose symmetry in artificial environments constructed to test research ideas—for example, the Trading Agent Competition (TAC) (Wellman et al., 2001) market games—since an objective of the model structure is to facilitate interagent comparisons.

The relevance of symmetry in games stems in large part from the opportunity to exploit this property for computational advantage. Symmetry immediately supports more compact representation, and may often enable analytic methods specific to this structure, or algorithmic shortcuts leading to significantly more effective or efficient solution procedures. Such techniques, however, may presume or impose constraints on solutions, for example by limiting consideration to pure or symmetric equilibria. Our own studies have often focused on symmetric games, and our attempts to exploit this property naturally raise the question of whether the symmetry-specialized techniques excessively restrict the search for solutions.

Given a symmetric environment, we may well prefer to identify symmetric equilibria, as asymmetric behavior seems relatively unintuitive (Kreps, 1990), and difficult to explain in a one-shot interaction. Rosenschein and Zlotkin (1994) argue that symmetric equilibria may be especially desirable for automated agents, since programmers can then publish and disseminate strategies for copying, without need for secrecy.

Despite the common use of symmetric constructions in game-theoretic analyses, the literature has not extensively investigated the general properties of symmetric games. One noteworthy exception is Nash's original paper on equilibria in non-cooperative games (Nash, 1951), which (in addition to presenting the seminal equilibrium existence result) considers a general concept of symmetric strategy profiles and its implication for symmetry of equilibria.

In the next section, we present three results on existence of equilibria in symmetric games. The first two identify special cases of symmetric games that possess pure-strategy equilibria. The third—derived directly from Nash's original result—establishes that any finite symmetric game has a (generally mixed) symmetric equilibrium. The remainder of the paper considers how the general existence of symmetric equilibria can be exploited by search methods that inherently focus on such solutions.

2. Existence Theorems for Symmetric Games

We first give a definition of a general normal-form game, adapted from Definition 7.D.2 in Mas-Colell et al. (1995), followed by the definition of a symmetric game.

Definition 1 *For a game with I players, the normal-form representation specifies for each player i a set of strategies*

S_i (with $s_i \in S_i$) and a payoff function $u_i(s_1, \ldots, s_I)$ giving the von Neumann-Morgenstern utilities for the outcome arising from strategies $(s_1, \ldots, s_I)$. We denote a game by the tuple $[I, \{S_i\}, \{u_i()\}]$.

Definition 2 ***(Symmetric Game)*** *A normal-form game is* symmetric *if the players have identical strategy spaces ($S_1 = S_2 = \ldots = S_I = S$) and $u_i(s_i, s_{-i}) = u_j(s_j, s_{-j})$, for $s_i = s_j$ and $s_{-i} = s_{-j}$ for all $i, j \in \{1, \ldots, I\}$. ($s_{-i}$ denotes all the strategies in profile s except for s_i.) Thus we can write $u(t, s)$ for the utility to any player playing strategy t in profile s. We denote a symmetric game by the tuple $[I, S, u()]$.*

Finally, we refer to a strategy profile with all players playing the same strategy as a *symmetric profile*, or, if such a profile is a Nash equilibrium, a *symmetric equilibrium*.

Theorem 1 *A symmetric game with two strategies has an equilibrium in pure strategies.*

Proof. Let $S = \{1, 2\}$ and let $[i, I - i]$ denote the profile with $i \in \{0, \ldots, I\}$ players playing strategy 1 and $I - i$ playing strategy 2. Let $u_s^i \equiv u(s, [i, I - i])$ be the payoff to a player playing strategy $s \in S$ in the profile $[i, I - i]$. Define the boolean function $eq(i)$ as follows:

$$eq(i) \equiv \begin{cases} u_2^0 \geq u_1^1 & \text{if } i = 0 \\ u_1^I \geq u_2^{I-1} & \text{if } i = I \\ u_1^i \geq u_2^{i-1} \text{ \& } u_2^i \geq u_1^{i+1} & \text{if } i \in \{1 \ldots I-1\}. \end{cases}$$

In words, $eq(i) = \text{TRUE}$ when no unilateral deviation from $[i, I-i]$ is beneficial—i.e., when $[i, I-i]$ is a pure-strategy equilibrium.

Assuming the opposite of what we want to prove, $eq(i) = \text{FALSE}$ for all $i \in \{0, \ldots, I\}$. We first show by induction on i that $u_2^i < u_1^{i+1}$ for all $i \in \{0, \ldots, I-1\}$. The base case, $i = 0$, follows directly from $eq(0) = \text{FALSE}$. For the general case, suppose $u_2^k < u_1^{k+1}$ for some $k \in \{0, \ldots, I-2\}$. Since $eq(k+1) = \text{FALSE}$, we have $u_1^{k+1} < u_2^k$ or $u_2^{k+1} < u_1^{k+2}$. The first disjunct contradicts the inductive hypothesis, implying $u_2^{k+1} < u_1^{k+2}$, which concludes the inductive proof. In particular, $u_2^{I-1} < u_1^I$. But, $eq(I) = \text{FALSE}$ implies $u_1^I < u_2^{I-1}$, and we have a contradiction. Therefore $eq(i) = \text{TRUE}$ for some i. $\square$

Can we relax the sufficient conditions of Theorem 1? First, consider symmetric games with more than two strategies. Rock-Paper-Scissors (Table 1a) is a counterexample showing that the theorem no longer holds. It is a three-strategy symmetric game with no pure-strategy equilibrium. This is the case because, of the six pure-strategy profiles (RR, RP, RS, PP, PS, SS), none constitute an equilibrium.

Next, does the theorem apply to *asymmetric* two-strategy games? Again, no, as demonstrated by Matching Pennies (Table 1b). In Matching Pennies, both players simultaneously choose an action—heads or tails—and player 1 wins if the actions match and player 2 wins otherwise. Again, none of the pure-strategy profiles (HH,TT,HT,TH) constitute an equilibrium.

Finally, can we strengthen the conclusion of Theorem 1 to guarantee *symmetric* pure-strategy equilibria? A simple anti-coordination game (Table 1c) serves as a counterexample. In this game, each player receives 1 when the players choose different actions and 0 otherwise. The only pure-strategy equilibria are the profiles where the players choose different actions, i.e., asymmetric profiles.

We next consider other sufficient conditions for pure and/or symmetric equilibria in symmetric games. First we present a lemma establishing properties of the mapping from profiles to best responses. The result is analogous to Lemma 8.AA.1 in Mas-Colell et al. (1995), adapted to the case of symmetric games.

Lemma 2 *In a symmetric game $[I, S, u()]$ with S nonempty, compact, and convex and with $u(s_i, s_1, \ldots, s_I)$ continuous in $(s_1, \ldots, s_I)$ and quasi-concave in s_i, the best response correspondence*

$$\begin{aligned} b(s) &\equiv \arg\max_{t \in S} u(t, s) \\ &= \{\tau : u(\tau, s) = \max_{t \in S} u(t, s)\} \end{aligned}$$

is nonempty, convex-valued, and upper hemicontinuous.

Proof. Since $u(S, s)$ is the continous image of the compact set S, it is compact and has a maximum and so $b(s)$ is nonempty. $b(S)$ is convex because the set of maxima of a quasiconcave function ($u(\cdot, s)$) on a convex set (S) is convex. To show that $b(\cdot)$ is upper hemicontinuous, show that for any sequence $s^n \to s$ such that $s^n \in b(s^n)$ for all n, we have $s \in b(s)$. To see this, note that for all n, $u(s_i^n, s^n) \geq u(s_i', s^n)$ for all $s_i' \in S$. So by continuity of $u(\cdot)$, we have $u(s_i, s) \geq u(s_i', s)$. $\square$

We now show our second main result, that infinite symmetric games with certain properties have symmetric equilibria in pure strategies. This result corresponds to Proposition 8.D.3 in Mas-Colell et al. (1995) which establishes the existence of (possibly asymmetric) pure strategy equilibria for the corresponding class of possibly asymmetric games.

Theorem 3 *A symmetric game $[I, S, u()]$ with S a nonempty, convex, and compact subset of some Euclidean space and $u(s_i, s_1, \ldots, s_I)$ continuous in $(s_1, \ldots, s_I)$ and quasiconcave in s_i has a symmetric pure-strategy equilibrium.*

Proof. $b(\cdot)$ is a correspondence from the nonempty, convex, compact set S to itself. By Lemma 2, $b(\cdot)$ is a nonempty,

	R	P	S
R	0,0	-1,1	1,-1
P	1,-1	0,0	-1,1
S	-1,1	1,-1	0,0

(a)

	H	T
H	1,0	0,1
T	0,1	1,0

(b)

	A	B
A	0,0	1,1
B	1,1	0,0

(c)

Table 1. (a) Rock-Paper-Scissors: a 3-player symmetric game with no pure-strategy equilibria. (b) Matching Pennies: a 2-player asymmetric game with no pure-strategy equilibria. (c) Anti-coordination Game: a symmetric game with no symmetric pure equilibria.

convex-valued, and upper hemicontinous correspondence. Thus, by Kakutani's Theorem[1], there exists a fixed point of $b(\cdot)$ which implies that there exists an $s \in S$ such that $s \in b(s)$ and so all playing s is a symmetric equilibrium. □

Finally, we present Nash's (1951) result that finite symmetric games have symmetric equilibria.[2] This is a special case of his result that every finite game has a "symmetric" equilibrium, where Nash's definition of a symmetric profile is one invariant under every automorphism of the game. This turns out to be equivalent to defining a symmetric profile as one in which all the symmetric players (if any) are playing the same mixed strategy. In the case of a symmetric game, the two notions of a symmetric profile (invariant under automorphisms vs. simply homogeneous) coincide and we have our result.

Here we present an alternate proof of the result, modeled on Nash's seminal proof of the existence of (possibly asymmetric) equilibria for general finite games.

Theorem 4 *A finite symmetric game has a symmetric mixed-strategy equilibrium.*

Proof. For each pure strategy $s \in S$, define a continuous function of a mixed strategy σ by

$$g_s(\sigma) \equiv \max(0, u(s, \sigma) - u(\sigma, \sigma)).$$

In words, $g_s(\sigma)$ is the gain, if any, of unilaterally deviating from the symmetric mixed profile of all playing σ to playing pure strategy s. Next define

$$y_s(\sigma) \equiv \frac{\sigma_s + g_s(\sigma)}{1 + \sum_{t \in S} g_t(\sigma)}.$$

The set of functions $y_s(\cdot) \forall s \in S$ defines a mapping from the set of mixed strategies to itself. We first show that the fixed points of $y(\cdot)$ are equilibria. Of all the pure strategies in the support of σ, one, say w, must be worst, implying $u(w, \sigma) \leq u(\sigma, \sigma)$ which implies that $g_w(\sigma) = 0$.

Assume $y(\sigma) = \sigma$. Then y must not decrease σ_w. The numerator is σ_w so the denominator must be 1, which implies that for all $s \in S$, $g_s(\sigma) = 0$ and so all playing σ is an equilibrium.

Conversely, if all playing σ is an equilibrium then all the g's vanish, making σ a fixed point under $y(\cdot)$.

Finally, since $y(\cdot)$ is a continuous mapping of a compact, convex set, it has a fixed point by Brouwer's Theorem. □

1 For this and other fixed point theorems (namely Brouwer's, used in Theorem 4) see, for example, the mathematical appendix (p952) of Mas-Colell et al. (1995).

2 This result can also be proved as a corollary to Theorem 3 by viewing a finite game as an infinite game with strategy sets being possible mixtures of pure strategies.

3. Computation of Symmetric Equilibria in Symmetric Games

Having affirmed the existence of symmetric equilibria in symmetric games, we consider next the computational implications for game solving. The methods considered here apply to finding one or a set of Nash equilibria (not all equilibria or equilibria satifying given properties).

3.1. Existing Game Solvers and Symmetry

A symmetric N-player game with S strategies has $\binom{N+S-1}{N}$ different profiles. Without exploiting symmetry, the payoff matrix requires S^N cells. This entails a huge computational cost just to store the payoff matrix. For example, for a 5-player game with 21 strategies, the payoff matrix needs over four million cells compared with 53 thousand when symmetry is exploited.

GAMBIT (McKelvey et al., 1992), the state-of-the-art tool for solving finite games, employs various algorithms (McKelvey and McLennan, 1996), none of which exploit symmetry. In our experience (Reeves et al., to appear; MacKie-Mason et al., 2004; Wellman et al., 2004), this renders GAMBIT unusable on many games of interest for which methods that do exploit symmetry yield results. We discuss two such methods in the next section.

3.2. Solving Games by Function Minimization

One of the many characterizations (McKelvey and McLennan, 1996) of a symmetric Nash equilibrium is as a global minimum of the following function from mixed strategies to the reals:

$$f(p) = \sum_{s \in S} \max[0, u(s,p) - u(p,p)]^2,$$

where $u(x,p)$ is the payoff from playing strategy x against everyone else playing strategy p. The function f is bounded below by zero and in fact for any equilibrium p, $f(p)$ is zero. This is because $f(p)$ is positive iff any pure strategy is a strictly better response than p itself.

This means that we can search for symmetric equilibria in symmetric games using any available function minimization technique. In particular, we can search for the root of f using the Amoeba algorithm (Press et al., 1992), a procedure for nonlinear function minimization based on the Nelder-Mead method (Nelder and Mead, 1965). In our previous work (Reeves et al., to appear) we have used an adaptation of Amoeba developed by Walsh et al. (2002) for finding symmetric mixed-strategy equilibria in symmetric games.

3.3. Solving Games by Replicator Dynamics

In his original exposition of the concept, Nash (1950) suggested an evolutionary interpretation of the Nash equilibrium. That idea can be codified as an algorithm for finding equilibria by employing the *replicator dynamics* formalism, introduced by Taylor and Jonker (1978) and Schuster and Sigmund (1983). Replicator dynamics refers to a process by which a population of strategies—where population proportions of the pure strategies correspond to mixed strategies—evolves over generations[3] by iteratively adjusting strategy populations according to performance with respect to the current mixture of strategies in the population. When this process reaches a fixed point, every pure strategy that hasn't died out is performing equally well given the current strategy mixture. Hence the final strategy mix in the population corresponds (under certain conditions (Reeves et al., to appear)) to a symmetric mixed strategy equilibrium. Although a more general form of replicator dynamics can be applied to asymmetric games (Gintis, 2000), it is particularly suited to searching for symmetric equilibria in symmetric games.

To implement this approach, we choose an initial population proportion for each pure strategy and then update them in successive generations so that strategies that perform well increase in the population at the expense of low-performing strategies. The proportion $p_g(s)$ of the population playing strategy s in generation g is given by

$$p_g(s) \propto p_{g-1}(s) \cdot (EP_s - W),$$

where EP_s is the expected payoff for pure strategy s against $N-1$ players all playing the mixed strategy corresponding to the population proportions, and W is a lower bound on payoffs (e.g., the minimum value in the payoff matrix) which serves as a dampening factor. To calculate the expected payoff EP_s from the payoff matrix, we average the payoffs for s in the profiles consisting of s concatenated with every profile of $N-1$ other agent strategies, weighted by the probabilities of the other agents' profiles. The probability of a particular profile $(n_1, \ldots n_S)$ of N agents' strategies, where n_s is the number of players playing strategy s, is

$$\frac{N!}{n_1! \cdot \ldots \cdot n_S!} \cdot p(1)^{n_1} \cdots p(S)^{n_S}.$$

(This is the multinomial coefficient multiplied by the probability of a profile if order mattered.)

We verify directly that the population update process has converged to a Nash equilibrium by checking that the evolved strategy is a best response to itself.

4. Discussion

A great majority of existing solution methods for games do not exploit symmetry. Consequently, many symmetric games are often practically unsolvable using these methods, while easily solvable using methods that do exploit symmetry and restrict the solution space to symmetric equilibria.

We have presented some general results about symmetric equilibria in symmetric games and discussed solution methods that exploit the existence result.

Since exploitation of symmetry has been known to produce dramatic simplifications of optimization problems (Boyd, 1990) it is somewhat surprising that it has not been actively addressed in the game theory literature. In addition to the methods we have used for game solving and that we discuss in Section 3 (replicator dynamics and function minimization with Amoeba), we believe symmetry may be productively exploited for other algorithms as well. For example, the well-known Lemke-Howson algorithm for 2-player (bimatrix) games lends itself to a simplification if the game is symmetric and if we restrict our attention to symmetric equilibria: we need only one payoff matrix, one mixed strategy vector, and one slackness vector, reducing the size of the Linear Complementarity Problem by a factor of two. As we discuss in Section 3.1, exploiting symmetry in games with more than

3 Instead of using discrete generations, the process can be defined continuously by a system of differential equations.

two players can yield enormous computational savings.

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