Deception in Finitely Repeated Security Games

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Abstract

Allocating resources to defend targets from attack is often complicated by uncertainty about the attacker's capabilities, objectives, or other underlying characteristics. In a repeated interaction setting, the defender can collect attack data over time to reduce this uncertainty and learn an effective defense. However, a clever attacker can manipulate the attack data to mislead the defender, influencing the learning process toward its own benefit. We investigate strategic deception on the part of an attacker with private type information, who interacts repeatedly with a defender. We present a detailed computation and analysis of both players' optimal strategies given the attacker may play deceptively. Computational experiments illuminate conditions conducive to strategic deception, and quantify benefits to the attacker. By taking into account the attacker's deception capacity, the defender can significantly mitigate loss from misleading attack actions.

Introduction

Real-world security domains are often characterized by *imperfect information*: uncertainty (particularly on the defender's part) about actions taken or underlying characteristics of the opposing agent. Experience observed through repeated interaction in such domains provides an opportunity for the defender to *learn* about the behaviors and characteristics of attacker(s) [Kar et al., 2017, Gholami et al., 2017, Haghtalab et al., 2016, Nguyen et al., 2016, Xu et al., 2016, Balcan et al., 2015, Blum et al., 2014, Marecki et al., 2012, Letchford et al., 2009]. For example, in wildlife protection [Fang et al., 2016], repeated interaction with poachers allows the defense authorities to observe poaching signs and patterns over time. From these observations, the defender may infer features of the poacher's capabilities and preferences, and thus design more effective patrolling strategies.

To the extent that the defender relies on data, however, the attacker may choose to modify its behavior to mislead the defender. That is, in a particular interaction the attacker may select an action that does not actually yield the best immediate reward, to avoid revealing sensitive private information. Such *deceptive* behavior could manipulate the outcome of learning to the long-term benefit of the attacker. A savvy defender, therefore, would take into account the attacker's manipulative strategy in designing his own strategy. We label an attacker strategy as *non-manipulative* if it chooses an action without regard to the defender's learning, for example if its behavior in each stage is a myopic best response. Previous work on learning in security games has generally treated the attacker as non-manipulative in this sense [Blum et al., 2014, Marecki et al., 2012].

We study the strategic deployment of attacker deception in finitely repeated security games. We adopt an incomplete-information model, where the defender has underlying uncertainty about the attacker's type. At each time step, the defender updates his belief on the attacker's type based on attack data collected at previous steps. Based on the updated belief, the defender chooses an action to play. The attacker decides its own action, aware that the defender is collecting attack data to infer about the attacker's type. The ultimate goal of both players is to maximize expected utility accumulated over the whole time horizon. A pair of strategies that best-respond to each other, accounting for observations, constitutes a Perfect Bayesian Nash Equilibrium (PBNE).

The paper includes four main contributions. First, we present a non-linear optimization program to find a PBNE of the finitely repeated simultaneous-move game. Second, we present a result that provides an easy technique to find a sequential equilibrium of the game based on a computed PBNE. Third, we show that there exists a PBNE in which players' equilibrium strategies depend only on histories of the attacker's actions. This allows us to represent both players' strategies in a compact form, which helps in significantly speeding up the equilibrium computation of the game. Fourth, we provide a preliminary extension to the Stackelberg game (sequential move) setting.

Finally, we present a detailed experimental analysis of strategic deception, showing how various game factors affect the tendency for the attacker to deviate from myopic best responses to mislead the defender. Our results show that the defender and attacker receive significant loss and benefit respectively if the defender does not address the attacker's deception. By taking into account deceptive attacks, such loss and benefit is reduced drastically.

Related Work

Learning in security games. Most existing work on learning in security games follows a Stackelberg model and assumes the attacker plays myopically at every time step [Kar et al., 2017, Gholami et al., 2017, Haghtalab et al., 2016, Nguyen et al., 2016, Blum et al., 2014, Marecki et al., 2012, Letchford et al., 2009]. Balcan et al. [2015] and Xu et al. [2016] study the problem of learning with no prior knowledge of the attacker's behavior. They take a regret-minimization approach to determine the defender's strategies at each time step.

Secrecy and deception in security games. Previous work studies security scenarios in which information available to the defender and attacker is asymmetric [Guo et al., 2017, Xu et al., 2015, Rabinovich et al., 2015, Hendricks and McAfee, 2006, Brown et al., 2005, Farrell and Rabin, 1996, Zhuang et al., 2010]. The defender can exploit that information asymmetricity to strategically reveal or disguise his information to the attacker. This results in responses of the attacker which are in favor of the defender. For example, in the model of Guo et al. [2017], the defender can disguise defense resources to deceive the attacker about the defender's type. We study an opposite scenario in which the attacker acts deceptively to mislead the defender.

Repeated games with incomplete information. Previous work has studied infinitely repeated games with incomplete information [Sorin, 2002, Aumann and Maschler, 1995, Jordan, 1995, Zamir, 1992, Forges, 1988]. These studies analyze properties and the convergence of players' strategies in an infinitely repeated game setting. We study the problem of one-sided incomplete information (i.e., uncertainty in the attacker's type) in finitely repeated security games.

Adversarial machine learning. There have been several studies on adversarial machine learning, attempting to investigate different attack scenarios on machine learning algorithms [Brückner et al., 2012, Brückner and Scheffer, 2011, Barreno et al., 2010, 2006, Lowd and Meek, 2005]. For example, *causative* attacks alters the training process by influencing the training data or *exploratory* attacks attempts to discover information about the learner and its training data. Different machine learning algorithms are then proposed which can resist these sophisticated attacks. Our work focuses on a causative attack scenario in security games. We aim at obtaining effective defense strategies which minimizes the damage of deceptive attacks in security games, given some learning outcome of attack data.

Game Model

In a finitely repeated *simultaneous-move* security game, there is a set of N targets, denoted by $\mathbf{N} = \{1, \dots, N\}$. A defender attempts to protect these targets by allocating limited security resources over these targets. Conversely, an attacker aims at attacking these targets. We denote by $K < N$ the number of the defender's security resources. At each time step t in a finite time horizon $\mathbf{T} = \{1, \dots, T\}$, both the defender and the attacker has to decide on which action to take. An action of the defender, $\mathbf{s}$, is an allocation of K resources over $\mathbf{N}$. We denote by $\mathbf{S}$ the set of all feasible actions of the defender. An action of the attacker is a target to attack. There is a set of attacker types $\mathbf{\Lambda} = \{1, \dots, L\}$. Each type $\lambda \in \mathbf{\Lambda}$ has a prior probability $p^\lambda \in (0, 1)$ such that $\sum_\lambda p^\lambda = 1$. At the beginning, Nature randomly draws a type to play the game according to a prior distribution $\{p^\lambda\}_\lambda$. The attacker knows its type while the defender does not. The defender is aware of $\{p^\lambda\}_\lambda$.

Player payoffs. Each target $i \in \mathbf{N}$ is associated with rewards and penalties of the defender, $\big(R^d(i), P^d(i)\big)$, and the attacker, $\big(R^\lambda(i), P^\lambda(i)\big)$, for every type $\lambda \in \mathbf{\Lambda}$. When the attacker of type $\lambda \in \mathbf{\Lambda}$ attacks i, if the defender is protecting i, the attacker receives a penalty $P^\lambda(i)$ while the defender obtains a reward $R^d(i)$. Conversely, if the defender is not protecting target i, the attacker gets $R^\lambda(i) > P^\lambda(i)$ while the defender receives $P^d(i) < R^d(i)$.

Player observations. At $t+1 \in \mathbf{T}$, both players observe their actions at previous time steps $\mathbf{h}_t = \{(\mathbf{s}_1, i_1), \dots, (\mathbf{s}_t, i_t)\}$ where $\mathbf{s}_{t'}$ and $i_{t'}$ are the defender and the attacker actions respectively at time step t'. We denote by $\mathbf{H}_t$ the set of all possible histories of length t and $\mathbf{H} = \{\mathbf{H}_t\}$ (where $t = 0, \dots, T-1$) the set of all histories. In particular, $\mathbf{H}_0 = \emptyset$. We denote by $\mathbf{h}^a_t = \{i_1, \dots, i_t\}$ a history of the attacker's actions and $\mathbf{H}^a_t$ the set of all these attack histories.

Behavioral strategies. At each step $t+1$, given a history $\mathbf{h}_t \in \mathbf{H}_t$, a behavioral strategy of the defender is a probability distribution $\mathbf{x}(\mathbf{h}_t) = \{x(\mathbf{s} \mid \mathbf{h}_t) : \sum_\mathbf{s} x(\mathbf{s} \mid \mathbf{h}_t) = 1, x(\mathbf{s} \mid \mathbf{h}_t) \in [0,1], \forall \mathbf{s} \in \mathbf{S}\}$ over the defender's action set $\mathbf{S}$. $x(\mathbf{s} \mid \mathbf{h}_t)$ is the probability the defender takes action $\mathbf{s} \in \mathbf{S}$ given the history $\mathbf{h}_t$. Similarly, a behavioral strategy of the attacker of type λ is a probability distribution $\mathbf{y}^\lambda(\mathbf{h}_t) = \{y^\lambda(i \mid \mathbf{h}_t) : \sum_i y^\lambda(i \mid \mathbf{h}_t) = 1, y^\lambda(i \mid \mathbf{h}_t) \in [0,1], \forall i \in \mathbf{N}\}$ over the attacker's actions $\mathbf{N}$. $y^\lambda(i \mid \mathbf{h}_t)$ is the probability the attacker of type λ attacks target i given $\mathbf{h}_t$. We denote by $\mathbf{x} = \{\mathbf{x}(\mathbf{h}_t)\}$ and $\mathbf{y}^\lambda = \{\mathbf{y}^\lambda(\mathbf{h}_t)\}$ strategies of the defender and attacker of type λ over all $\mathbf{h}_t \in \mathbf{H}$ respectively. Finally, $\mathbf{X}$ and $\mathbf{Y} = \{\mathbf{Y}^\lambda\}$ denote the sets of all strategies $\mathbf{x}$ and $\mathbf{y}^\lambda$ respectively.

Player expected utilities. Let $\mathbf{x}$ and $\mathbf{y} = \{\mathbf{y}^\lambda\}$ be the defender and attacker's behavioral strategies respectively. At each $t+1$, the defender can update his belief on the attacker types using the Bayes rule, which is formulated as:

$$p(\lambda \mid \mathbf{h}_t) \propto p^\lambda \prod\nolimits_{t'=1}^{t} y^\lambda(i_{t'} \mid \mathbf{h}_{t'-1})$$

where $\mathbf{h}_0 = \emptyset, \mathbf{h}_{t'} = \{\mathbf{h}_{t'-1}, i_{t'}\}$. Let:

$$EU^d_i(\mathbf{x}, \mathbf{h}_t) = \Big[\sum\nolimits_{\mathbf{s}: i \in \mathbf{s}} x(\mathbf{s} \mid \mathbf{h}_t)\Big] \big(R^d(i) - P^d(i)\big) + P^d(i)$$

$$EU^\lambda_i(\mathbf{x}, \mathbf{h}_t) = \Big[\sum\nolimits_{\mathbf{s}: i \in \mathbf{s}} x(\mathbf{s} \mid \mathbf{h}_t)\Big] \big(P^\lambda(i) - R^\lambda(i)\big) + R^\lambda(i)$$

be immediate expected utilities of the defender and the attacker of type λ respectively at target i at step $t+1$ given the defender plays $\mathbf{x}(\mathbf{h}_t)$. Based on the immediate expected utilities at every target, the players' total expected utilities over $\mathbf{T}$ can be computed using backward induction as follows.

At the last time step T, the total expected utilities of the defender and the attacker with respect to history $\mathbf{h}_{T-1} \in \mathbf{H}_{T-1}$ is equal to their immediate expected utilities at $\mathbf{h}_{T-1}$:

$$U^d_T(\mathbf{x}, \mathbf{y} \mid \mathbf{h}_{T-1}) = \sum_{\lambda, i} p(\lambda \mid \mathbf{h}_{T-1}) y^\lambda(i \mid \mathbf{h}_{T-1}) EU^d_i(\mathbf{x}, \mathbf{h}_{T-1})$$

$$U^\lambda_T(\mathbf{x}, \mathbf{y} \mid \mathbf{h}_{T-1}) = \sum\nolimits_i y^\lambda(i \mid \mathbf{h}_{T-1}) EU^\lambda_i(\mathbf{x}, \mathbf{h}_{T-1}), \forall \lambda$$

At time step $t+1 < T$, the total expected utilities of both players with respect to history $\mathbf{h}_t \in \mathbf{H}_t$ consists of (i) the immediate expected utility at $t+1$; and (ii) the future expected utility after $t+1$. These utilities are formulated as:

$$U^d_{t+1}(\mathbf{x},\mathbf{y} \mid \mathbf{h}_t) = \sum\nolimits_{\lambda,i} p(\lambda \mid \mathbf{h}_t) y^\lambda(i \mid \mathbf{h}_t) EU^d_i(\mathbf{x},\mathbf{h}_t)$$
$$+ \sum\nolimits_{\mathbf{s},\lambda,i} x(\mathbf{s} \mid \mathbf{h}_t) p(\lambda \mid \mathbf{h}_t) y^\lambda(i \mid \mathbf{h}_t) U^d_{t+2}(\mathbf{x},\mathbf{y} \mid \mathbf{h}_t,(\mathbf{s},i))$$
$$U^\lambda_{t+1}(\mathbf{x},\mathbf{y} \mid \mathbf{h}_t) = \sum\nolimits_{i} y^\lambda(i \mid \mathbf{h}_t) EU^\lambda_i(\mathbf{x},\mathbf{h}_t)$$
$$+ \sum\nolimits_{\mathbf{s},i} x(\mathbf{s} \mid \mathbf{h}_t) y^\lambda(i \mid \mathbf{h}_t) U^\lambda_{t+2}(\mathbf{x},\mathbf{y} \mid \mathbf{h}_t,(\mathbf{s},i)), \forall\lambda$$

Player's goals. Given any history $\mathbf{h}_t$, both players aim at choosing strategies $\mathbf{x}(\mathbf{h}_t), \{\mathbf{y}^\lambda(\mathbf{h}_t)\}$ that maximize their total expected utility at $\mathbf{h}_t$. In this scenario, the attacker is no longer myopic; it has to reason about all future possibilities to decide on which behavioral strategy to play at each $\mathbf{h}_t$. Such attack strategies (which may not be myopically optimal) are chosen to mislead the defender about the attacker's type, ultimately benefiting the attacker in future steps. These optimal behavioral strategies of players form a PBNE.

Game Equilibria

Definition 1 (PBNE). *Behavioral strategies of the defender* $\mathbf{x}^*$ *and attacker* $\mathbf{y}^*$ *form a PBNE of the game if and only if for every* $\mathbf{h}_t \in \mathbf{H}_t$ *that occurs, we have:*

- $\mathbf{x}^*$ *is the best response of the defender:*
$$U^d_{t+1}(\mathbf{x}^*,\mathbf{y}^* \mid \mathbf{h}_t) \geq U^d_{t+1}(\mathbf{x},\mathbf{y}^* \mid \mathbf{h}_t), \forall \mathbf{x} \in \mathbf{X}$$
- $\mathbf{y}^{\lambda,*}$ *is the best response of the attacker type* λ*:*
$$U^\lambda_{t+1}(\mathbf{x}^*,\mathbf{y}^* \mid \mathbf{h}_t) \geq U^\lambda_{t+1}(\mathbf{x}^*,\mathbf{y} \mid \mathbf{h}_t), \forall \mathbf{y} \in \mathbf{Y}$$

Since the action sets of both players are finite, there always exists a PBNE of the game. Our first result extends a given PBNE to a refined sequential equilibrium.

Theorem 1. *For each PBNE, there is a sequential equilibrium in which players' strategies are identical to the ones in the PBNE at histories that occur with a positive probability.*

Proof. We denote by $(\mathbf{x},\mathbf{y})$ a PBNE of the game. We follow the trembling-hand approach to find a corresponding sequential equilibrium of $(\mathbf{x},\mathbf{y})$. Let $r^d(\mathbf{h}_t)$ and $r^\lambda(\mathbf{h}_t)$ be the ratios of the number of zero probabilities to the number of non-zero probabilities in $\mathbf{x}(\mathbf{h}_t)$ and $\mathbf{y}^\lambda(\mathbf{h}_t)$ respectively. For each $\epsilon > 0$, we construct a new fully mixed behavioral strategy of the defender and the attacker, $(\mathbf{x}_\epsilon,\mathbf{y}_\epsilon)$, as follows:

$$x_\epsilon(\mathbf{s} \mid \mathbf{h}_t) = \epsilon, \text{ if } x(\mathbf{s} \mid \mathbf{h}_t) = 0$$
$$x_\epsilon(\mathbf{s} \mid \mathbf{h}_t) = x(\mathbf{s} \mid \mathbf{h}_t) - \epsilon * r^d(\mathbf{h}_t), \text{ if } x(\mathbf{s} \mid \mathbf{h}_t) > 0$$
$$y^\lambda_\epsilon(i \mid \mathbf{h}_t) = \epsilon, \text{ if } y^\lambda(i \mid \mathbf{h}_t) = 0$$
$$y^\lambda_\epsilon(i \mid \mathbf{h}_t) = y^\lambda(i \mid \mathbf{h}_t) - \epsilon * r^\lambda(\mathbf{h}_t), \text{ if } y^\lambda(i \mid \mathbf{h}_t) > 0$$

ϵ is chosen to be small enough such that all resulting probabilities are positive. We use the breath-first search (according to the time horizon $\mathbf{T}$) to examine the whole history set. When encountering a history with a zero probability of occurrence $p(\mathbf{h}_t) = 0$ according to $(\mathbf{x},\mathbf{y})$, we construct a new belief of the defender over attacker types at $\mathbf{h}_t$ as follows:

$$p'(\lambda \mid \mathbf{h}_t) = \lim_{\epsilon\to 0} p_\epsilon(\lambda \mid \mathbf{h}_t) = \lim_{\epsilon\to 0} \frac{p^\lambda \prod_{t'} y^\lambda_\epsilon(i_{t'} \mid \mathbf{h}_{t'-1})}{\sum_{\lambda'} p^{\lambda'} \prod_{t'} y^{\lambda'}_\epsilon(i_{t'} \mid \mathbf{h}_{t'-1})}$$

We find a PBNE of the corresponding sub-game starting from this history $\mathbf{h}_t$ with this new belief. We then replace the strategies of the sub-game in $(\mathbf{x},\mathbf{y})$ with these new equilibrium strategies. We also update $(\mathbf{x}_\epsilon,\mathbf{y}_\epsilon)$ accordingly with the updated strategies of the sub-game. This process will continue until all histories are examined. The resulting strategies $(\mathbf{x}',\mathbf{y}')$ with belief $\{p'(\lambda \mid \mathbf{h}_t)\}$ belong to a sequential equilibrium of the game. Indeed, it is straightforward to prove:

$$x'(\mathbf{s} \mid \mathbf{h}_t) = \lim_{\epsilon\to 0} x_\epsilon(\mathbf{s} \mid \mathbf{h}_t)$$
$$y^{\lambda,'}(i \mid \mathbf{h}_t) = \lim_{\epsilon\to 0} y^\lambda_\epsilon(i \mid \mathbf{h}_t)$$
$$p'(\lambda \mid \mathbf{h}_t) = \lim_{\epsilon\to 0} p_\epsilon(\lambda \mid \mathbf{h}_t) \text{ (by definition)}$$

Furthermore, the updating process only replaces strategies in $(\mathbf{x},\mathbf{y})$ at histories $\mathbf{h}_t$ with a zero-probability of occurrence by a PBNE of the sub-game at $\mathbf{h}_t$ respective to the belief $p'(\lambda \mid \mathbf{h}_t)$. Therefore, $(\mathbf{x}',\mathbf{y}')$ is the best response of the players at every $\mathbf{h}_t$ according to $p'(\lambda \mid \mathbf{h}_t)$. □

Next, we present a result that enables a compact representation of the game. We denote by $\mathbf{X}^a$ a subset of behavioral strategies of the defender in which all the strategies are independent of histories of the defender's actions. In other words, for all $\mathbf{x} \in \mathbf{X}^a$, $x(\mathbf{s} \mid \mathbf{h}_t) = x(\mathbf{s} \mid \mathbf{h}^a_t)$ for every history $\mathbf{h}_t$ where $\mathbf{h}^a_t$ is the corresponding history of attacker actions. Similarly, $\mathbf{Y}^a$ is a subset of behavioral strategies of the attacker.

Theorem 2. *There exists a PBNE of the game in which the equilibrium strategies of the players only depend on the histories of the attacker's actions.*

Proof. We use Brouwer's fixed-point theorem and a backward induction method. We are going to show that there exists a PBNE $\mathbf{x}^* \in \mathbf{X}^a$ and $\mathbf{y}^* \in \mathbf{Y}^a$ such that: $\forall \mathbf{h}_t$

$$U^d_{t+1}(\mathbf{x}^*,\mathbf{y}^* \mid \mathbf{h}_t) \geq U^d_{t+1}(\mathbf{x},\mathbf{y}^* \mid \mathbf{h}_t), \forall \mathbf{x} \in \mathbf{X} \quad (1)$$
$$U^\lambda_{t+1}(\mathbf{x}^*,\mathbf{y}^* \mid \mathbf{h}_t) \geq U^\lambda_{t+1}(\mathbf{x}^*,\mathbf{y} \mid \mathbf{h}_t), \forall \mathbf{y} \in \mathbf{Y} \quad (2)$$

We denote by $\mathbf{x}\{\mathbf{x}(\mathbf{h}_t) \leftarrow \mathbf{s}\}$ the defender strategy obtained by replacing $\mathbf{x}(\mathbf{h}_t)$ in $\mathbf{x}$ by a defense action $\mathbf{s}$. Similarly, $\mathbf{y}^\lambda\{\mathbf{y}^\lambda(\mathbf{h}_t) \leftarrow i\}$ is the attacker strategy of type λ obtained by replacing $\mathbf{y}^\lambda(\mathbf{h}_t)$ in $\mathbf{y}^\lambda$ by an attack action i. We define:

$$\phi^d_\mathbf{s}(\mathbf{x},\mathbf{y} \mid \mathbf{h}_t) = \max\{0, U^d_{t+1}(\mathbf{x}\{\mathbf{x}(\mathbf{h}_t) \leftarrow \mathbf{s}\},\mathbf{y} \mid \mathbf{h}_t) - U^d_{t+1}(\mathbf{x},\mathbf{y} \mid \mathbf{h}_t)\}$$
$$\phi^\lambda_i(\mathbf{x},\mathbf{y} \mid \mathbf{h}_t) = \max\{0, U^\lambda_{t+1}(\mathbf{x},\mathbf{y}\{\mathbf{y}^\lambda(\mathbf{h}_t) \leftarrow i\}) \mid \mathbf{h}_t) - U^\lambda_{t+1}(\mathbf{x},\mathbf{y} \mid \mathbf{h}_t)\}$$

which are non-negative continuous functions in $(\mathbf{x},\mathbf{y})$. We define a function $F : (\mathbf{X}^a,\mathbf{Y}^a) \to (\mathbf{X}^a,\mathbf{Y}^a)$ as follows: $F(\mathbf{x},\mathbf{y}) = (\mathbf{x}',\mathbf{y}')$ where

$$x'(\mathbf{s} \mid \mathbf{h}_t) = \frac{x(\mathbf{s} \mid \mathbf{h}_t) + \phi^d_\mathbf{s}(\mathbf{x},\mathbf{y} \mid \mathbf{h}_t)}{\sum_{\mathbf{s}'} x(\mathbf{s}' \mid \mathbf{h}_t) + \phi^d_{\mathbf{s}'}(\mathbf{x},\mathbf{y} \mid \mathbf{h}_t)}$$
$$y^{\lambda,'}(i \mid \mathbf{h}_t) = \frac{y^\lambda(i \mid \mathbf{h}_t) + \phi^\lambda_i(\mathbf{x},\mathbf{y} \mid \mathbf{h}_t)}{\sum_j y^\lambda(j \mid \mathbf{h}_t) + \phi^\lambda_j(\mathbf{x},\mathbf{y} \mid \mathbf{h}_t)}$$

Since F is continuous over a convex and compact set $(\mathbf{X}^a, \mathbf{Y}^a)$, there exists $(\mathbf{x}^*, \mathbf{y}^*)$ such that $F(\mathbf{x}^*, \mathbf{y}^*) = (\mathbf{x}^*, \mathbf{y}^*)$ according to the Brouwer's fixed point theorem. On the other hand, according to the linearity of expectation, there must be an action $\mathbf{s}$ such that $U^d_{t+1}(\mathbf{x}^*\{\mathbf{x}^*(\mathbf{h}_t) \leftarrow \mathbf{s}\}, \mathbf{y}^* \mid \mathbf{h}_t) - U^d_{t+1}(\mathbf{x}^*, \mathbf{y}^* \mid \mathbf{h}_t) \leq 0$, meaning that $\phi^d_{\mathbf{s}}(\mathbf{x}, \mathbf{y} \mid \mathbf{h}_t) = 0$. Therefore, we have:

$$x^*(\mathbf{s} \mid \mathbf{h}_t) = \frac{x^*(\mathbf{s} \mid \mathbf{h}_t)}{1 + \sum_{\mathbf{s}'} \phi^d_{\mathbf{s}'}(\mathbf{x}^*, \mathbf{y}^* \mid \mathbf{h}_t)}$$

This implies $\phi^d_{\mathbf{s}'}(\mathbf{x}^*, \mathbf{y}^* \mid \mathbf{h}_t) = 0, \forall \mathbf{s}'$. Similarly, $\phi^\lambda_i(\mathbf{x}^*, \mathbf{y}^* \mid \mathbf{h}_t) = 0, \forall i$. As a result, we obtain:

$$U^d_{t+1}(\mathbf{x}^*\{\mathbf{x}^*(\mathbf{h}_t) \leftarrow \mathbf{s}\}, \mathbf{y}^* \mid \mathbf{h}_t) \leq U^d_{t+1}(\mathbf{x}^*, \mathbf{y}^* \mid \mathbf{h}_t) \quad (3)$$
$$U^\lambda_{t+1}(\mathbf{x}^*, \mathbf{y}^*\{\mathbf{y}^{\lambda,*}(\mathbf{h}_t) \leftarrow i\}) \mid \mathbf{h}_t) \leq U^\lambda_{t+1}(\mathbf{x}^*, \mathbf{y}^* \mid \mathbf{h}_t) \quad (4)$$

for all $\mathbf{s}$ and i and $\mathbf{h}_t$. While the above result holds for any convex and compact $(\mathbf{X}^a, \mathbf{Y}^a)$, showing that the fixed point profile provides more utility than any deviation to $\mathbf{X}$ or $\mathbf{Y}$ requires $(\mathbf{X}^a, \mathbf{Y}^a)$ to depend on the attacker's past actions. This dependence is due to the dependence of the posterior belief on the attacker's past actions, as can be seen in the next steps of the proof. Based on the above inequalities, we show that $(\mathbf{x}^*, \mathbf{y}^*)$ satisfy (1–2) using backward induction.

At last time step T, for every $\mathbf{x} \in \mathbf{X}$, we have:

$$\begin{aligned} & U^d_T(\mathbf{x}^*, \mathbf{y}^* \mid \mathbf{h}_{T-1}) \\ & \geq \sum_{\mathbf{s}} x(\mathbf{s} \mid \mathbf{h}_{T-1}) U^d_T(\mathbf{x}^*\{\mathbf{x}^*(\mathbf{h}_{T-1}) \leftarrow \mathbf{s}\}, \mathbf{y}^* \mid \mathbf{h}_{T-1}) \\ & = U^d_T(\mathbf{x}, \mathbf{y}^* \mid \mathbf{h}_{T-1}) \end{aligned}$$

Therefore, $\mathbf{x}^*$ is the defender best response against the attacker's strategy $\mathbf{y}^*$ at time step T.

At time step $t+1 < T$, suppose that (1–2) hold true for all $t' > t+1$, then for every $\mathbf{x} \in \mathbf{X}$, we have:

$$\begin{aligned} & U^d_{t+1}(\mathbf{x}^*, \mathbf{y}^* \mid \mathbf{h}_t) \\ & \geq \sum_{\mathbf{s}} x(\mathbf{s} \mid \mathbf{h}_t) U^d_{t+1}(\mathbf{x}^*\{\mathbf{x}^*(\mathbf{h}_t) \leftarrow \mathbf{s}\}, \mathbf{y}^* \mid \mathbf{h}_t) \\ & = \sum_{\lambda,i} p(\lambda \mid \mathbf{h}_t) y^{\lambda,*}(i \mid \mathbf{h}_t) EU^d_i(\mathbf{x}, \mathbf{h}_t) + \sum_{\mathbf{s}} x(\mathbf{s} \mid \mathbf{h}_t) \\ & \qquad \times \sum_{i,\lambda} p(\lambda \mid \mathbf{h}_t) y^{\lambda,*}(i \mid \mathbf{h}_t) U^d_{t+2}(\mathbf{x}^*, \mathbf{y}^* \mid \mathbf{h}_t, (\mathbf{s}, i)) \\ & \geq \sum_{\lambda,i} p(\lambda \mid \mathbf{h}_t) y^{\lambda,*}(i \mid \mathbf{h}_t) EU^d_i(\mathbf{x}, \mathbf{h}_t) + \sum_{\mathbf{s}} x(\mathbf{s} \mid \mathbf{h}_t) \\ & \qquad \times \sum_{i,\lambda} p(\lambda \mid \mathbf{h}_t) y^{\lambda,*}(i \mid \mathbf{h}_t) U^d_{t+2}(\mathbf{x}, \mathbf{y}^* \mid \mathbf{h}_t, (\mathbf{s}, i)) \\ & = U^d_{t+1}(\mathbf{x}, \mathbf{y}^* \mid \mathbf{h}_t) \end{aligned}$$

Therefore, $\mathbf{x}^*$ is the defender best response against the attacker's strategy $\mathbf{y}^*$ at $t+1$. Similarly, $\mathbf{y}^*$ is the attacker's best response against $\mathbf{x}^*$ at all time steps. $\square$

Equilibrium Computation

Based on Theorem 2, in computing a PBNE, we only need to search over the strategy sets $(\mathbf{X}^a, \mathbf{Y}^a)$. We also only need to consider attack histories $\{\mathbf{h}^a_t\}$. We can now represent the defender behavioral strategies as compact marginal coverage probabilities over targets. We overload the notation $\mathbf{x}(\mathbf{h}^a_t) = \{x(i \mid \mathbf{h}^a_t)\}$ where $x(i \mid \mathbf{h}^a_t)$ is the defender's coverage probability at target i at history $\mathbf{h}^a_t$ such that $\sum_i x(i \mid \mathbf{h}^a_t) \leq K$ and $x(i \mid \mathbf{h}^a_t) \in [0,1]$ for all $i \in \mathbf{N}$. In particular, $x(i \mid \mathbf{h}^a_t) = \sum_{\mathbf{s}: i \in \mathbf{s}} x(\mathbf{s} \mid \mathbf{h}^a_t)$. The players' immediate and total expected utilities can be reformulated accordingly as follows:

$$EU^d_i(\mathbf{x}, \mathbf{h}^a_t) = x(i \mid \mathbf{h}^a_t)(R^d(i) - P^d(i)) + P^d(i) \quad (5)$$
$$EU^\lambda_i(\mathbf{x}, \mathbf{h}^a_t) = x(i \mid \mathbf{h}^a_t)(P^\lambda(i) - R^\lambda(i)) + R^\lambda(i) \quad (6)$$
$$\begin{aligned} U^d_{t+1}(\mathbf{x}, \mathbf{y} \mid \mathbf{h}^a_t) &= \sum_{\lambda,i} p(\lambda \mid \mathbf{h}^a_t) y^\lambda(i \mid \mathbf{h}^a_t) EU^d_i(\mathbf{x}, \mathbf{h}^a_t) \\ & + \sum_{i,\lambda} p(\lambda \mid \mathbf{h}^a_t) y^\lambda(i \mid \mathbf{h}^a_t) U^d_{t+2}(\mathbf{x}, \mathbf{y} \mid \mathbf{h}^a_t, i) \quad (7) \end{aligned}$$
$$\begin{aligned} U^\lambda_{t+1}(\mathbf{x}, \mathbf{y} \mid \mathbf{h}^a_t) &= \sum_i y^\lambda(i \mid \mathbf{h}^a_t) EU^\lambda_i(\mathbf{x}, \mathbf{h}^a_t) \\ & + \sum_i y^\lambda(i \mid \mathbf{h}^a_t) U^\lambda_{t+2}(\mathbf{x}, \mathbf{y} \mid \mathbf{h}^a_t, i) \quad (8) \end{aligned}$$

Note that the total expected utilities of players at last time step $U^d_T(\mathbf{x}, \mathbf{y} \mid \mathbf{h}^a_{T-1})$ and $U^\lambda_T(\mathbf{x}, \mathbf{y} \mid \mathbf{h}^a_{T-1})$ do not have the second term (which represents the future expected utility) as in Equations (7–8). In the following, we present a backward-induction based method to find the attacker (defender) best response against a fixed behavioral strategy of the defender (attacker). We then introduce a program to compute a PBNE based on these best-response solutions.

Attacker best response

Given a defender's strategy $\mathbf{x}$, we can compute a best response of the attacker of type λ using backward induction:

At last time step T. Given a history $\mathbf{h}^a_{T-1}$, finding a best response of the attacker type λ against $\mathbf{x}$ is formulated as the following linear program:

$$\max_{\mathbf{y}^\lambda(\mathbf{h}^a_{T-1})} \sum_i y^\lambda(i \mid \mathbf{h}^a_{T-1}) EU^\lambda_i(\mathbf{x}, \mathbf{h}^a_{T-1}) \quad (9)$$
$$\text{s.t.} \sum_i y^\lambda(i \mid \mathbf{h}^a_{T-1}) = 1, y^\lambda(i \mid \mathbf{h}^a_{T-1}) \geq 0, \forall i \quad (10)$$

which maximizes the attacker's total expected utility at $\mathbf{h}^a_{T-1}$. Its corresponding dual program:

$$\min_{v^\lambda(\mathbf{h}^a_{T-1})} v^\lambda(\mathbf{h}^a_{T-1}) \quad (11)$$
$$\text{s.t. } v^\lambda(\mathbf{h}^a_{T-1}) \geq EU^\lambda_i(\mathbf{x}, \mathbf{h}^a_{T-1}), \forall i \in \mathbf{N}. \quad (12)$$

According to complementary slackness, any optimal primal and dual solutions $(\mathbf{y}^{\lambda,*}(\mathbf{h}^a_{T-1}), v^\lambda_*(\mathbf{h}^a_{T-1}))$ satisfies: $\forall i$

$$y^{\lambda,*}(i \mid \mathbf{h}^a_{T-1}) \left[v^\lambda_*(\mathbf{h}^a_{T-1}) - EU^\lambda_i(\mathbf{x}, \mathbf{h}^a_{T-1}) \right] = 0 \quad (13)$$

At time step $t+1 < T$. Given a history $\mathbf{h}^a_t$, finding a best response of the attacker type λ is formulated as:

$$\max_{\mathbf{y}^\lambda(\mathbf{h}^a_t)} \sum_i y^\lambda(i \mid \mathbf{h}^a_t) \left[EU^\lambda_i(\mathbf{x}, \mathbf{h}^a_t) + v^\lambda_*(\mathbf{h}^a_t, i) \right] \quad (14)$$
$$\text{s.t.} \sum_i y^\lambda(i \mid \mathbf{h}^a_t) = 1, y^\lambda(i \mid \mathbf{h}^a_t) \geq 0, \forall i \in \mathbf{N}. \quad (15)$$

which maximizes the attacker's total expected utility at $\mathbf{h}_t^a$ where $v_*^\lambda(\mathbf{h}_t^a, i)$ is the attacker's optimal total expected utility at $(\mathbf{h}_t^a, i)$. Its corresponding dual program:

$$\min_{v^\lambda(\mathbf{h}_t^a)} v^\lambda(\mathbf{h}_t^a) \quad (16)$$

$$\text{s.t. } v^\lambda(\mathbf{h}_t^a) \geq EU_i^\lambda(\mathbf{x}, \mathbf{h}_t^a) + v_*^\lambda(\mathbf{h}_t^a, i), \forall i \quad (17)$$

According to complementary slackness: any optimal solution $(\mathbf{y}^{\lambda,*}(\mathbf{h}_t^a), v_*^\lambda(\mathbf{h}_t^a))$ must satisfy: $\forall i$:

$$y^{\lambda,*}(i \mid \mathbf{h}_t)\left[v_*^\lambda(\mathbf{h}_t) - EU_i^\lambda(\mathbf{x}, \mathbf{h}_t) - v_*^\lambda(\mathbf{h}_t, i)\right] = 0 \quad (18)$$

Defender best response

Given an attack strategy $\mathbf{y}$, we denote by:

$$\bar{p}(\lambda \mid \mathbf{h}_t^a) = p^\lambda \prod_{t'=1}^{t} y^\lambda(i_{t'} \mid \mathbf{h}_{t'-1}^a)$$

Then the defender's belief on attacker types at each history $\mathbf{h}_t^a$ can be computed as follows:

$$p(\lambda \mid \mathbf{h}_t^a) = \frac{\bar{p}(\lambda \mid \mathbf{h}_t^a)}{\sum_{\lambda'} \bar{p}(\lambda' \mid \mathbf{h}_t^a)} \quad (19)$$

Similar to the computation of a best response of attacker, we can compute a best response of the defender against an attack strategy $\mathbf{y}$ using backward induction as follows:

At last time step T. Given a history $\mathbf{h}_{T-1}^a$, finding a best response $\mathbf{x}(\mathbf{h}_{T-1}^a)$ can be formulated as:

$$\max \sum_\lambda \bar{p}(\lambda \mid \mathbf{h}_{T-1}^a) \sum_i y^\lambda(i \mid \mathbf{h}_{T-1}) EU_i^d(\mathbf{x}, \mathbf{h}_{T-1}^a) \quad (20)$$

$$\text{s.t. } \sum_i x(i \mid \mathbf{h}_{T-1}) \leq K, x(i \mid \mathbf{h}_{T-1}) \in [0,1], \forall i. \quad (21)$$

Proposition 1. *For every attack history $\mathbf{h}_{T-1}^a$, we denote by $v_*^d(\mathbf{h}_{T-1}^a)$ the defender's optimal total expected utility against the attacker's strategy $\mathbf{y}$ at $\mathbf{h}_{T-1}^a$. Then:*

$$v_*^d(\mathbf{h}_{T-1}^a) = \frac{\bar{v}_*^d(\mathbf{h}_{T-1}^a)}{\sum_{\lambda'} \bar{p}(\lambda' \mid \mathbf{h}_{T-1}^a)}$$

The proof of Proposition 1 is in the appendix.[1] Here, $\bar{v}_*^d(\mathbf{h}_{T-1}^a)$ the optimal objective of (20–21). By removing the constant $P^d(i)$ in $EU_i^d(\mathbf{x}, \mathbf{h}_{T-1}^a)$ (Equation 5) and taking the dualty, we obtain the corresponding dual program:

$$\min\ K\bar{v}^d(0 \mid \mathbf{h}_{T-1}^a) + \sum_i \bar{v}^d(i \mid \mathbf{h}_{T-1}^a) \quad (22)$$

$$\text{s.t. } \bar{v}^d(0 \mid \mathbf{h}_{T-1}^a) \geq 0, \bar{v}^d(i \mid \mathbf{h}_{T-1}^a) \geq 0, \forall i \quad (23)$$

$$\bar{v}^d(0 \mid \mathbf{h}_{T-1}^a) + \bar{v}^d(i \mid \mathbf{h}_{T-1}^a) \geq \quad (24)$$
$$\sum_\lambda \bar{p}(\lambda \mid \mathbf{h}_{T-1}^a) y^\lambda(i \mid \mathbf{h}_{T-1}^a)\left(R^d(i) - P^d(i)\right), \forall i.$$

According to complementary slackness, any optimal solutions $(\mathbf{x}^*(\mathbf{h}_{T-1}^a), \{\bar{v}_*^d(i \mid \mathbf{h}_{T-1}^a)\}, \bar{v}_*^d(0 \mid \mathbf{h}_{T-1}^a))$ satisfies: $\forall i$

$$x^*(i \mid \mathbf{h}_{T-1}^a)[\bar{v}_*^d(0 \mid \mathbf{h}_{T-1}^a) + \bar{v}_*^d(i \mid \mathbf{h}_{T-1}^a) \quad (25)$$
$$- \sum_\lambda \bar{p}(\lambda \mid \mathbf{h}_{T-1}^a) y^\lambda(i \mid \mathbf{h}_{T-1}^a)\left(R^d(i) - P^d(i)\right)] = 0$$

$$\bar{v}_*^d(i \mid \mathbf{h}_{T-1}^a)\left[x^*(i \mid \mathbf{h}_{T-1}^a) - 1\right] = 0 \quad (26)$$

$$\bar{v}_*^d(0 \mid \mathbf{h}_{T-1}^a)\left[\sum_j x^*(j \mid \mathbf{h}_{T-1}^a) - K\right] = 0 \quad (27)$$

[1] Link: https://ix.cs.uoregon.edu/~thanhhng/publications/ConfPaper/AAAI19_Appendix.pdf

At time step $t+1 < T$. Given a history $\mathbf{h}_t^a$, finding an optimal behavioral strategy $\mathbf{x}(\mathbf{h}_t^a)$ can be formulated as the following program:

$$\max_{\mathbf{x}(\mathbf{h}_t^a)} \sum_{\lambda,i} \bar{p}(\lambda \mid \mathbf{h}_t^a) y^\lambda(i \mid \mathbf{h}_t^a) EU_i^d(\mathbf{x}, \mathbf{h}_t^a) \quad (28)$$
$$+ \sum_i \bar{v}_*^d(\mathbf{h}_t^a, i)$$

$$\text{s.t. } \sum_i x(i \mid \mathbf{h}_t^a) \leq K, x(i \mid \mathbf{h}_t^a) \in [0,1], \forall i \quad (29)$$

We denote by $\bar{v}_*^d(\mathbf{h}_t^a)$ the optimal objective of (28–29) at $\mathbf{h}_t$. In (28), $\bar{v}_*^d(\mathbf{h}_t^a, i)$ is the optimal objective of this primal program (28–29) but with respect to the history $(\mathbf{h}_t^a, i)$.

Proposition 2. *For every attack history $\mathbf{h}_t^a$, we denote by $v_*^d(\mathbf{h}_t^a)$ the defender's optimal total expected utility against the attacker's strategy $\mathbf{y}$ at $\mathbf{h}_t^a$. Then:*

$$v_*^d(\mathbf{h}_t^a) = \frac{\bar{v}_*^d(\mathbf{h}_t^a)}{\sum_{\lambda'} \bar{p}(\lambda' \mid \mathbf{h}_t^a)}$$

The proof of Proposition 2 is in the appendix. In (28), the term $\sum_i \bar{v}_*^d(\mathbf{h}_t^a, i)$ and the term $P^d(i)$ in $EU_i^d(\mathbf{x}, \mathbf{h}_t^a)$ are constant. By removing these constants and taking the dual, we obtain the corresponding dual program:

$$\min K\bar{v}^d(0 \mid \mathbf{h}_t^a) + \sum_i \bar{v}^d(i \mid \mathbf{h}_t^a) \quad (30)$$

$$\text{s.t. } \bar{v}^d(0 \mid \mathbf{h}_t^a) \geq 0, \bar{v}^d(i \mid \mathbf{h}_t^a) \geq 0, \forall i \in \mathbf{N} \quad (31)$$

$$\bar{v}^d(0 \mid \mathbf{h}_t^a) + \bar{v}^d(i \mid \mathbf{h}_t^a) \geq \quad (32)$$
$$\sum_\lambda \bar{p}(\lambda \mid \mathbf{h}_t) y^\lambda(i \mid \mathbf{h}_t^a)\left(R^d(i) - P^d(i)\right), \forall i.$$

According to complementary slackness, any optimal solution $(\mathbf{x}^*(\mathbf{h}_t^a), \{\bar{v}_*^d(i \mid \mathbf{h}_t^a)\}, \bar{v}_*^d(0 \mid \mathbf{h}_t^a))$ satisfies: $\forall i$:

$$x^*(i \mid \mathbf{h}_t^a)\left[\bar{v}_*^d(0 \mid \mathbf{h}_t^a) + \bar{v}_*^d(i \mid \mathbf{h}_t^a) - \quad (33)$$
$$\sum_\lambda \bar{p}(\lambda \mid \mathbf{h}_t^a) y^\lambda(i \mid \mathbf{h}_t^a)\left(R^d(i) - P^d(i)\right)\right] = 0$$

$$\bar{v}_*^d(i \mid \mathbf{h}_t^a)\left[x^*(i \mid \mathbf{h}_t^a) - 1\right] = 0 \quad (34)$$

$$\bar{v}_*^d(0 \mid \mathbf{h}_t^a)\left[\sum_i x^*(i \mid \mathbf{h}_t^a) - K\right] = 0 \quad (35)$$

Equilibrium computation program

Based on the computation of players' best responses, a pair of behavioral strategies $(\mathbf{x}, \mathbf{y})$ forms a PBNE if and only if these strategies satisfy (i) the feasibility constraints (21,23,24,29,31,32) and (10,12,15,17); and (ii) the complementary slackness constraints (25–27, 33–35) and (13, 18). Since finding strategies which satisfy these slackness constraints is not straightforward, we convert the problem of finding a PBNE into the following program:

$$\min \delta \quad \text{such that } \forall i, \mathbf{h}_t^a: \quad (36)$$

$$\delta \geq y^\lambda(i \mid \mathbf{h}_t^a)\left[v^\lambda(\mathbf{h}_t^a) - EU_i^\lambda(\mathbf{x}, \mathbf{h}_t^a) - v^\lambda(\mathbf{h}_t^a, i)\right] \quad (37)$$

$$\delta \geq x(i \mid \mathbf{h}_t^a)\left[\bar{v}^d(0 \mid \mathbf{h}_t^a) + \bar{v}^d(i \mid \mathbf{h}_t^a) \quad (38)$$
$$- \sum_\lambda \bar{p}(\lambda \mid \mathbf{h}_t^a) y^\lambda(i \mid \mathbf{h}_t^a)\left(R^d(i) - P^d(i)\right)\right]$$

$$\delta \geq \bar{v}^d(i \mid \mathbf{h}_t^a)\left[x(i \mid \mathbf{h}_t^a) - 1\right] \quad (39)$$

$$\delta \geq \bar{v}^d(0 \mid \mathbf{h}_t^a)\left[\sum_i x(i \mid \mathbf{h}_t^a) - K\right] \quad (40)$$

Constraints (21,23,24,29,31,32), (10,12,15,17) (41) where $v^{\lambda}(\mathbf{h}^a_t, i) = 0$ if $t = T-1$. Constraints (37) and (38–40) correspond to the complementary slackness constraints of the attacker and defender respectively. Note that any equilibrium of the game is a feasible solution of the program (36–41) which returns an objective value of $\delta = 0$. On the other hand, the right-hand side of constraints (37–40) is always non-negative due to constraint (41). Thus, $\delta \geq 0$ for all feasible solutions of the program (36–41). It means that any equilibrium of the game is an optimal solution of (36–41). In addition, since the optimal objective value $\delta = 0$, any optimal solution of (36–41) returns a value of zero for all the right-hand sides of (37–40). Therefore, any optimal solution of this program is a PBNE.

Extension to Stackelberg Setting

In the Stackelberg game model, a mixed strategy of the defender is defined as a probability distribution $\mathbf{m} = \{\mathbf{m}(\mathbf{s}) : \sum_{\mathbf{s}} \mathbf{m}(\mathbf{s}) = 1, \mathbf{m}(\mathbf{s}) \in [0,1]\}$ over the action set $\mathbf{S}$. We denote by $\mathbf{M}$ the set of all mixed strategies of the defender. At each time step, the defender commits to a mixed strategy. The attacker is aware of that mixed strategy and then decides which target to attack. Therefore, in finitely repeated Stackelberg games, at each time step $t+1$, an observation of the defender is a history $\mathbf{h}_t = \{(\mathbf{m}_1, i_1), \ldots, (\mathbf{m}_t, i_t)\}$ while an observation of the attacker is a history $(\mathbf{h}_t, \mathbf{m}_{t+1})$.

The behavioral strategy of the defender at $\mathbf{h}_t$ is a probability distribution $\mathbf{x}(\mathbf{h}_t) = \{x(\mathbf{m} \mid \mathbf{h}_t) : \sum_{\mathbf{m}} x(\mathbf{m} \mid \mathbf{h}_t) = 1, x(\mathbf{m} \mid \mathbf{h}_t) \in [0,1]\}$ over the set of mixed strategies of the defender. On the other hand, a behavioral strategy of the attacker of type λ at $(\mathbf{h}_t, \mathbf{m}_{t+1})$ is a probability distribution $\mathbf{y}^{\lambda}(\mathbf{h}_t, \mathbf{m}_{t+1}) = \{y^{\lambda}(i \mid \mathbf{h}_t, \mathbf{m}_{t+1}) : \sum_i y^{\lambda}(i \mid \mathbf{h}_t, \mathbf{m}_{t+1}) = 1, y^{\lambda}(i \mid \mathbf{h}_t, \mathbf{m}_{t+1}) \in [0,1]\}$. A PBNE of Stackelberg security games is then defined similarly as simultaneous-move games. Since the set of mixed strategies of the defender is infinite, the existence of a PBNE in Stackelberg security games is an open research question. Nevertheless, we can compute an ϵ-PBNE by discretizing this set of defense mixed strategies and applying the same backward induction method as in the simultaneous case.

We specifically analyze the deception of the attacker in finitely repeated Stackelberg security games with $|\mathbf{N}| = 2$, $|\mathbf{\Lambda}| = 2$, and $K = 1$. We adopt the tradition in Stackelberg security game that rewards and penalties are strictly positive and negative respectively for both players. We consider a game scenario in which the defender only plays a *pure* behavioral strategy in $\mathbf{X}^{\text{pure}} = \{\mathbf{x} : x(\mathbf{m} \mid \mathbf{h}_t) = 1, \text{ for some } \mathbf{m} \in \mathbf{M}, \forall \mathbf{h}_t\}$.

Theorem 3. *In a finitely repeated Stackelberg security game with $|\mathbf{N}| = 2$, $|\mathbf{\Lambda}| = 2$, and $K = 1$, if the defender only plays a pure behavioral strategy in $\mathbf{X}^{pure}$ and the rewards and penalties are strictly positive and negative respectively for both the players, there exists a PBNE of the game in which the attacker plays a myopic best response at every history $(\mathbf{h}_t, \mathbf{m}_{t+1})$.*

One significance of this preliminary result is that the assumption about a myopic attacker in previous work on finitely repeated Stackelberg security games is justified (at least in the simple setting of this result) even when the attackers care about future expected utility. In future research, we aim to generalize this special case and explore the deception patterns for multiple targets and multiple types in the Stackelberg setting.

Experiments

We focus on the attacker's strategic use of deception. In our experiments, the players' rewards and penalties are generated uniformly at random in the range $[1, 10]$ and $[-10, -1]$ respectively.

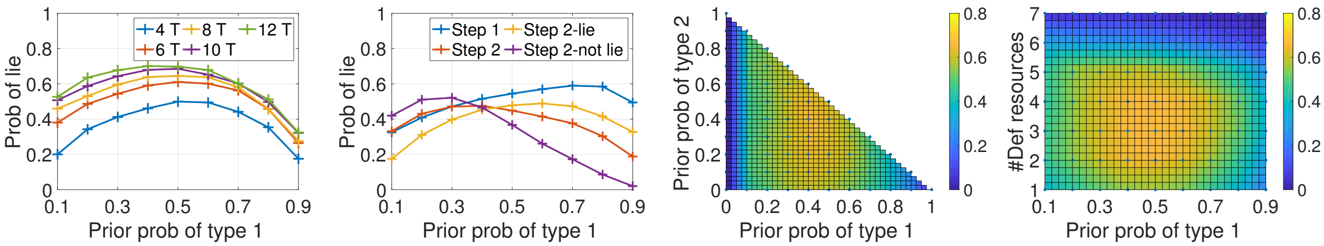
Analysis of attacker deception

The purpose of deception is to shift the defender's belief away from the attacker's true type. Any action on part of the attacker toward this purpose must take into account similar reasoning by other attacker types. Further, shaping the belief of the defender is beneficial only if it results in a later gain for the attacker. In the following, we present our results with respect to an attacker of type 1. The behavior for other attacker types is symmetric.

In our first experiment, we analyze games with number of attacker types: $|\mathbf{\Lambda}| = 2$, number of targets: $|\mathbf{N}| \in \{4, 6, 8, 10, 12\}$, and number of time steps $|\mathbf{T}| \in \{2, 3\}$. Results are shown in Figure 1(a)(b). The x-axis is the prior probability of attacker type 1. The y-axis is the probability a type-1 attacker attacks a myopically non-optimal target (i.e., probability of deceptive action, or "lie" for short) at time step $t = 1$ or $t = 2$ (for 3-step games). Each data point is averaged over 220 game instances. Figure 1(a) shows results for 2-step games; each curve corresponds to a number of targets. Overall, the attacker's tendency to deceive is roughly concave in the prior probability of its true type. This makes sense, as deception has relatively less power to change the beliefs of a defender when they start near extremes. We also see an increase in deception with the number of targets. This reflects the growth in options for deception, as well as increased potential benefit for misleading the defender.

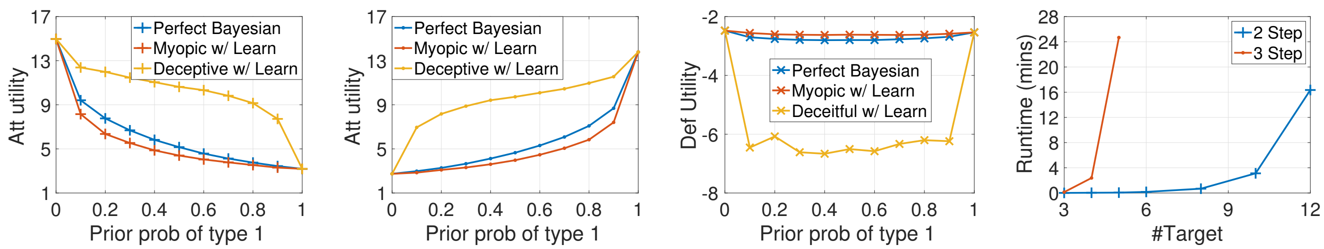
Results for 3-step games are shown in Figure 1(b). We present deception probabilities for the attacker of type 1 at: (1) Step 1; (ii) Step 2; (iii) Step 2-lie (step 2 conditioned that the attacker lied at step 1); and (iv) Step 2-not lie (step 2 conditioned that the attacker did not lie at step 1). In this figure, $|\mathbf{N}| = 4$. As for the 2-step game, the probability of deception in each case is roughly concave in the prior. The probability of deception at step 1 is somewhat elevated in the 3-step game, since the attacker accrues longer-term benefit from misleading the defender. Moreover, the peak is shifted to the right, reflecting increased chance for successful deception given its opportunity to repeat the lie over two periods.

Indeed, given that the attacker lies at step 1, the attacker lies with roughly proportional probability at step 2 (blue curve versus yellow curve). On the other hand, when the attacker does not lie at step 1, its pattern of deception at step 2 (purple curve) is qualitatively different. Switching to be deceptive at step 2 is more promising at low priors (where the act has some chance of misleading), and very unlikely at high priors where there is little chance to mislead the defender if it had not already started in step 1.



(a) 2-type, 2-step games (b) 2-type, 3-step games (c) 3-type, 2-step games (d) 2-type, 2-step games

Figure 1: Attacker deception analysis, attacker type 1.



(a) Attacker type 1 (b) Attacker type 2 (c) Defender (d) Runtime

Figure 2: Solution comparison and runtime performance.

In our second experiment, we analyze deceptive strategies of the attacker type 1 in 2-step games with the number of attacker types is $|\mathbf{\Lambda}| = 3$. In these games, $|\mathbf{N}| = 8$. The result is shown in Figure 1(c). The x-axis and y-axis represent prior probabilities of types 1 and 2 respectively. Figure 1(c) shows that the attacker deception tendency is unimodal with respect the prior of its type 1, and less sensitive ot the distribution across other types.

In our third experiment, we vary the number of defender resources in 2-step games with 2 attacker types. The result is shown in Figure 1(d). When the number of defender resources is high (close to the number of the targets), the defender can provide a high coverage probability at all targets. Specially, when $K = |\mathbf{N}|$, the defender protects all targets all the time. As a result, the attacker may not achieve any benefit by lying. Therefore, the attacker lies less when K gets closer to $|\mathbf{N}|$.

Solution quality and runtime performance

In our last experiment, we compare the players' utilities for playing strategies computed in three scenarios:

1. Perfect Bayesian. The attacker is rationally deceptive and the defender takes into account the potential deceit.
2. Myopic w/ Learn. The attacker is myopic and the defender also assumes so.
3. Deceptive w/ Learn. The attacker is rationally deceptive while the defender assumes the attacker is myopic.

The defender performs a Bayesian update on his belief about the attacker's type in all three cases. Results are shown in Figures 2(a)(b)(c), averaging over 220 3-step game instances with two attacker types and five targets. The x-axis is the prior probability of type 1. The y-axis is the attacker utility of each type or the defender utility on average. Figures 2(a)(b)(c) show if the defender does not account for deception, the rationally deceptive attacker achieves a significant gain while the defender suffers a significant loss (yellow versus red curves). When the defender accounts for the prospect of deception of the attacker, such gains and losses are drastically reduced (blue versus red).

Finally, we display in Figure 2(d) the runtime performance of our equilibrium-finding algorithm. The x-axis is the number of targets and the y-axis is runtime in minutes. For 2-step games, the runtime remains modest for up to ten targets. For $|\mathbf{T}| = 3$, the runtime grows quickly and exceeds 24 minutes when the number of targets is five.

Summary

We study the problem of deception in finitely repeated security games. In these games, the defender collects attack data over time to learn about the attacker type while the attacker plays deceptively to mislead the defender. We present a detailed analysis and computation of finding optimal strategies of players in the games. We then show through computational experiments that the attacker (defender) receives a great benefit (loss) when the defender does not take into account deceptive attacks. Conversely, such benefit (loss) is reduced significantly when the defender addresses the attacker's deception.

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Appendix

Proof of Propositions 1&2

We are going to show that: for all $\mathbf{h}_t^a$

$$v_*^d(\mathbf{h}_t^a) = \frac{\bar{v}_*^d(\mathbf{h}_t^a)}{\sum_{\lambda'} \bar{p}(\lambda' \mid \mathbf{h}_t^a)} \tag{42}$$

We follow the backward induction approach:

At last time step T, for every defender strategy $\mathbf{x}(\mathbf{h}_{T-1}^a)$, according to Equation 15, the objective in (16) can be reformulated as follows:

$$\bar{v}^d(\mathbf{h}_{T-1}^a) = \sum_\lambda \bar{p}(\lambda \mid \mathbf{h}_{T-1}^a) \sum_i y^\lambda(i \mid \mathbf{h}_{T-1}) EU_i^d(\mathbf{x}, \mathbf{h}_{T-1}^a)$$
$$= \left[\sum\nolimits_{\lambda'} \bar{p}(\lambda' \mid \mathbf{h}_{T-1}^a)\right] \times \sum\nolimits_{\lambda,i} p(\lambda \mid \mathbf{h}_{T-1}^a) y^\lambda(i \mid \mathbf{h}_{T-1}) EU_i^d(\mathbf{x}, \mathbf{h}_{T-1}^a)$$
$$= \left[\sum\nolimits_{\lambda'} \bar{p}(\lambda' \mid \mathbf{h}_{T-1}^a)\right] U_T^d(\mathbf{x}, \mathbf{y} \mid \mathbf{h}_{T-1}^a)$$

Therefore, Equation 42 is true for all $\mathbf{h}_{T-1}^a$.

At time step $t+1$, suppose that Equation (42) holds true for all $\mathbf{h}_{t'}$ with $t' \geq t+1$, we are going to show that it is true for all history $\mathbf{h}_t$. Indeed, for every defense strategy $\mathbf{x}(\mathbf{h}_t^a)$, the objective (24) can be reformulated as follows:

$$\bar{v}^d(\mathbf{h}_t^a) = \sum\nolimits_{\lambda,i} \bar{p}(\lambda \mid \mathbf{h}_t^a) y^\lambda(i \mid \mathbf{h}_t^a) EU_i^d(\mathbf{x}, \mathbf{h}_t^a) + \sum\nolimits_i \bar{v}_*^d(\mathbf{h}_t^a, i)$$
$$= \left(\sum_{\lambda'} \bar{p}(\lambda' \mid \mathbf{h}_t^a)\right) \sum_{\lambda,i} p(\lambda \mid \mathbf{h}_t^a) y^\lambda(i \mid \mathbf{h}_t^a) EU_i^d(\mathbf{x}, \mathbf{h}_t^a) + \sum\nolimits_i \left[\sum\nolimits_\lambda \bar{p}(\lambda | \mathbf{h}_t^a, i)\right] v_*^d(\mathbf{h}_t^a, i)$$

The second term of RHS can be rewritten as follows:

$$\sum\nolimits_i \left[\sum\nolimits_\lambda \bar{p}(\lambda \mid \mathbf{h}_t^a, i)\right] v_*^d(\mathbf{h}_t^a, i)$$
$$= \sum\nolimits_i \left[\sum\nolimits_\lambda \bar{p}(\lambda | \mathbf{h}_t^a) y^\lambda(i | \mathbf{h}_t^a)\right] v_*^d(\mathbf{h}_t^a, i)$$
$$= \sum_i \left[\sum_\lambda \left(\sum_{\lambda'} \bar{p}(\lambda' | \mathbf{h}_t^a)\right) p(\lambda | \mathbf{h}_t^a) y^\lambda(i | \mathbf{h}_t^a)\right] v_*^d(\mathbf{h}_t^a, i)$$
$$= \left(\sum_{\lambda'} \bar{p}(\lambda' | \mathbf{h}_t^a)\right) \sum_i \left[\sum_\lambda p(\lambda | \mathbf{h}_t^a) y^\lambda(i | \mathbf{h}_t^a)\right] v_*^d(\mathbf{h}_t^a, i)$$

Therefore, we obtain:

$$\bar{v}_*^d(\mathbf{h}_t^a) = \left(\sum_{\lambda'} \bar{p}(\lambda' | \mathbf{h}_t^a)\right) v_*^d(\mathbf{h}_t^a)$$

which means Equation 42 is true for all $\mathbf{h}_t^a$.

Stackelberg Game Model

Our Stackelberg game model equilibrium computation is formulated as optimization problems in a setting with $\mathbf{T} = 2$, $|\mathbf{N}| = 2$, and $K = 1$. At the first time step, both players maximize their overall utility. At the second time step, they maximize their myopic utility. Specifically, their optimization problems are formed as follows.

For the attacker:

- At last time step $T = 2$, given an observation $\mathbf{m}_2$, an optimal behavioral strategy $\{y^\lambda(i|\mathbf{h}_1, \mathbf{m}_2)\}$ of the attacker of type λ is formulated as the following linear optimization problem:

$$\max_{\{y^\lambda(i|\mathbf{h}_1,\mathbf{m}_2)\}} \sum_i y^\lambda(i|\mathbf{h}_1, \mathbf{m}_2)[\mathbf{m}_2(i)\left(P^\lambda(i) - R^\lambda(i)\right) + R^\lambda(i)]$$
$$\text{s.t.} \sum_i y^\lambda(i|\mathbf{h}_1, \mathbf{m}_2) = 1$$
$$y^\lambda(i|\mathbf{h}_1, \mathbf{m}_2) \geq 0, \ \forall i \in \mathbf{N}.$$

  Since at the last time step the attacker always plays best response, its behavioral strategy only depends on the defender's strategy $\mathbf{m}_2$ he observed.

- At time step $t = 1$, given an observation $\mathbf{m}_1$, an optimal behavioral strategy $\{y^\lambda(i|\mathbf{m}_1)\}$ of the attacker of type λ is formulated as the following linear optimization problem:

$$\max_{\{y^\lambda(i|\mathbf{m}_1)\}} \sum_i y^\lambda(i|\mathbf{m}_1)[\mathbf{m}_1(i)\left(P^\lambda(i) - R^\lambda(i)\right) + R^\lambda(i)] + \sum_i y^\lambda(i|\mathbf{m}_1) v_*^\lambda(f(i, \mathbf{m}_1))$$
$$\text{s.t.} \sum_i y^\lambda(i|\mathbf{m}_1) = 1$$
$$y^\lambda(i|\mathbf{m}_1) \geq 0, \ \forall i \in \mathbf{N}.$$

  where $v_*^\lambda(\mathbf{m_2})$ is the unscaled future optimal utility of the attacker type λ given the observation $\mathbf{m}_2$. f is a function that chooses strategy for the defender given the observation i.

For the defender:

- At last time step $T = 2$, given observation $\mathbf{h}_1$, an optimal mixed strategy $\mathbf{m}_2$ is formulated as the following linear optimization problem:

$$\max_{\mathbf{m}_2} \sum_\lambda \bar{p}(\lambda|\mathbf{h}_1) EU^{d,\lambda}\left(\mathbf{m_2}, \{y^\lambda(i|\mathbf{h}_1, \mathbf{m_2})\}\right)$$
$$\text{s.t.} \sum_i \mathbf{m}_2(i) = 1$$
$$\mathbf{m}_2(i) \geq 0, \forall i \in N$$

  where

$$EU^{d,\lambda}\left(\mathbf{m_2}, \{y^\lambda(i|\mathbf{h}_1, \mathbf{m_2})\}\right) = \sum_i y^\lambda(i|\mathbf{h}_1, \mathbf{m_2})\left[\mathbf{m_2}(i)\left(R^d(i) - P^d(i)\right) + P^d(i)\right]$$

- At time step $t = 1$, an optimal mixed strategy $\mathbf{m}_1$ is formulated as the following linear optimization problem:

$$\max_{\mathbf{m}_1} \sum_{\lambda} p^{\lambda} EU^{d,\lambda}\left(\mathbf{m_1}, \{y^{\lambda}(i|\mathbf{m_1})\}\right) + \sum_{i} v_*^d(\mathbf{m_1}, i)$$
$$\text{s.t.} \sum_{i} \mathbf{m}_1(i) = 1$$
$$\mathbf{m}_1(i) \geq 0, \forall i \in N$$

where $\frac{v_*^d(\mathbf{m_1},i)}{\sum_{\lambda'} \bar{p}(\lambda'|\mathbf{m_1},i)}$ is the optimal expected utility of the defender given observation $(\mathbf{m_1}, i)$ and

$$EU^{d,\lambda}\left(\mathbf{m_1}, \{y^{\lambda}(i|\mathbf{m_1})\}\right) = \sum_{i} y^{\lambda}(i|\mathbf{m_1})\left[\mathbf{m_1}(i)\left(R^d(i) - P^d(i)\right) + P^d(i)\right]$$

If there exists a Bayesian equilibrium, the strategies in the equilibrium should satisfy those optimization problems simultaneously.

Proof of Theorem 3

Before proving Theorem 3, we present two preliminary results.

Lemma 1. *In a finitely repeated Stackelberg security game with $|\mathbf{N}| = 2$, $|\mathbf{\Lambda}| = 2$, and $K = 1$, given that both types of attackers play pure behavioral strategies, the set of the possible defender's equilibrium strategies at time step 2 is finite and strategies in this set are given by specific formulas.*

At the second time step, both types of attackers always play pure strategies. We denote U_λ^i as the type λ attacker's utility by attacking target i at the second time step, given the defender's strategy. We denote by $\mathbf{M}(\lambda) = (M(\lambda), 1 - M(\lambda))$ the defender's strategy that makes type λ attacker indifferent over the two targets at the second time step where $M(\lambda)$ is the coverage probability on target 1. We call defender's strategy $\mathbf{M}(\lambda)$ the "indifferent strategy" for the type λ attacker. We denote $\mathbf{m}_{max}$ ($\mathbf{m}_{min}$) as defender's strategy that makes the type of attackers who have larger (smaller) $M(\lambda)$ indifferent over the two targets. Denote $\mathbf{m}_t(i)$ as the coverage probability of defender's strategy on target i at time step t.

Specifically, we have

- $U_1^1 = \mathbf{m}_2(1)\left(P^1(1) - R^1(1)\right) + R^1(1)$
- $U_1^2 = \mathbf{m}_2(2)\left(P^1(2) - R^1(2)\right) + R^1(2)$
- $U_2^1 = \mathbf{m}_2(1)\left(P^2(1) - R^2(1)\right) + R^2(1)$
- $U_2^2 = \mathbf{m}_2(2)\left(P^2(2) - R^2(2)\right) + R^2(2)$
- $M(\lambda) = \frac{R^\lambda(1) - P^\lambda(2)}{R^\lambda(1) - P^\lambda(2) + R^\lambda(2) - P^\lambda(1)}$
- $\mathbf{m}_{max} = (\max_\lambda M(\lambda), 1 - \max_\lambda M(\lambda))$
- $\mathbf{m}_{min} = (\min_\lambda M(\lambda), 1 - \min_\lambda M(\lambda))$

Proof. First, we list all possible combinations of the attackers' best responses and the corresponding utility relationship without equality. We will discuss the equality scenarios later. Cases are indexed and these indices are consistent in the following tables.

Table 1: Attackers' best response

idx	$y^1(1\vert\mathbf{m}_2)$	$y^1(2\vert\mathbf{m}_2)$	$y^2(1\vert\mathbf{m}_2)$	$y^2(2\vert\mathbf{m}_2)$
1	0	1	0	1
2	1	0	1	0
3	1	0	0	1
4	0	1	1	0

Table 2: Utility Relationship

idx	Type 1	Type 2
1	$U_1^1 - U_1^2 < 0$	$U_2^1 - U_2^2 < 0$
2	$U_1^1 - U_1^2 > 0$	$U_2^1 - U_2^2 > 0$
3	$U_1^1 - U_1^2 > 0$	$U_2^1 - U_2^2 < 0$
4	$U_1^1 - U_1^2 < 0$	$U_2^1 - U_2^2 > 0$

We obtain the defender's strategy space(Table 3) by solving the inequalities in Table 2. For example, in the case 1, we have

$$U_1^1 - U_1^2 < 0$$
$$\mathbf{m}_2(1)\left(P^1(1) - R^1(1)\right) + R^1(1) - \mathbf{m}_2(2)\left(P^1(2) - R^1(2)\right) - R^1(2) < 0$$
$$\mathbf{m}_2(1)\left(P^1(1) - R^1(1) + P^1(2) - R^1(2)\right) + R^1(1) - P^1(2) < 0$$
$$\mathbf{m}_2(1)\left(-P^1(1) + R^1(1) - P^1(2) + R^1(2)\right) > R^1(1) - P^1(2)$$

$$\Rightarrow \mathbf{m}_2(1) > \frac{R^1(1) - P^1(2)}{-P^1(1) + R^1(1) - P^1(2) + R^1(2)} = M(1)$$

Thus, $\mathbf{m}_2(1) > M(1)$. Similarly, $\mathbf{m}_2(1) > M(2)$. So we have $\mathbf{m}_2(1) \in (\mathbf{m}_{max}(1), 1]$. Same reasoning process can be applied for other cases.

Table 3: Defender's strategy space

idx	Action Spaces	Condition
1	$\mathbf{m}_2(1) \in (\mathbf{m}_{max}(1), 1]$	
2	$\mathbf{m}_2(1) \in [0, \mathbf{m}_{min}(1))$	
3	$\mathbf{m}_2(1) \in (\mathbf{m}_{min}(1), \mathbf{m}_{max}(1))$	$M(1) > M(2)$
4	$\mathbf{m}_2(1) \in (\mathbf{m}_{min}(1), \mathbf{m}_{max}(1))$	$M(1) < M(2)$

Notice that for case 3 and case 4, only one of them could happen, which depends on the relationship between $M(1)$ and $M(2)$.

Second, we discuss the possible equilibrium strategies of the defender in each strategy space and show the defender's indifferent strategies for two types of attackers are those equilibrium strategies.

In an equilibrium, any player who deviates from the equilibrium on its own has nothing to gain. Based on this property, we disturb the defender's strategy and explore the existence of equilibrium strategies in each strategy space.

At the second time step, the defender's utility is given by $\sum_\lambda \bar{p}(\lambda|\mathbf{h}_1)EU^{d,\lambda}\left(\mathbf{m}_2, \{y^\lambda(i|\mathbf{m}_2)\}\right)$ as shown in Stackelberg model.

- **Action space 1**

$$\begin{aligned}
&\sum_\lambda \bar{p}(\lambda|\mathbf{h}_1)EU^{d,\lambda}\left(\mathbf{m}_2, \{y^\lambda(i|\mathbf{m}_2)\}\right)\\
&=\bar{p}(1|\mathbf{h}_1)\sum_i y^1(i|\mathbf{m}_2)[\mathbf{m}_2(i)(R^d(i)-P^d(i))+P^d(i)]\\
&+\bar{p}(2|\mathbf{h_1})\sum_i y^2(i|\mathbf{m}_2)[\mathbf{m}_2(j)(R^d(i)-P^d(i))+P^d(i)]\\
&=\bar{p}(1|\mathbf{h}_1)[\mathbf{m}_2(2)(R^d(2)-P^d(2))+P^d(2)]\\
&\quad+\bar{p}(2|\mathbf{h_1})[\mathbf{m}_2(2)(R^d(2)-P^d(2))+P^d(2)]\\
&=[\bar{p}(1|\mathbf{h_1})+\bar{p}(2|\mathbf{h_1})][\mathbf{m}_2(2)(R^d(2)-P^d(2))+P^d(2)]\\
&=[\bar{p}(1|\mathbf{h_1})+\bar{p}(2|\mathbf{h_1})][R^d(2)-\mathbf{m}_2(1)(R^d(2)-P^d(2))]
\end{aligned}$$

The above is decreasing with $\mathbf{m}_2(1)$. Given the above holds for $\mathbf{m}_2(1) \in (\mathbf{m}_{\max}(1), 1]$, we can conclude that the equilibrium does not lie in $\mathbf{m}_2(1) \in (\mathbf{m}_{\max}(1), 1]$, otherwise it is always possible to decrease $\mathbf{m}_2(1)$ to obtain a better utility. So $\mathbf{m}_2(1)$ will converge to $\mathbf{m}_{\max}(1)$.

- **Action space 2**

By the same reasoning process, the defender's utility at the second time step is increasing with $\mathbf{m}_2(1)$. Given the above holds for $\mathbf{m}_2(1) \in [0, \mathbf{m}_{\min}(1))$, we can conclude that the equilibrium does not lie in $\mathbf{m}_2(1) \in [0, \mathbf{m}_{\min}(1))$, otherwise it is always possible to increase $\mathbf{m}_2(1)$ to obtain a better utility. So $\mathbf{m}_2(1)$ will converge to $\mathbf{m}_{\min}(1)$.

- **Action space 3**

$$\begin{aligned}
&\sum_\lambda \bar{p}(\lambda|\mathbf{h}_1)EU^{d,\lambda}\left(\mathbf{m}_2, \{y^\lambda(i|\mathbf{m}_2)\}\right)\\
&=\bar{p}(1|\mathbf{h}_1)\sum_i y^1(i|\mathbf{m}_2)[\mathbf{m}_2(i)(R^d(i)-P^d(i))+P^d(i)]\\
&+\bar{p}(2|\mathbf{h_1})\sum_i y^2(i|\mathbf{m}_2)[\mathbf{m}_2(i)(R^d(i)-P^d(i))+P^d(i)]\\
&=\bar{p}(1|\mathbf{h_1})[\mathbf{m}_2(1)(R^d(1)-P^d(1))+P^d(1)]\\
&\quad+\bar{p}(2|\mathbf{h_1})[\mathbf{m}_2(2)(R^d(2)-P^d(2))+P^d(2)]
\end{aligned}$$

Assume $\mathbf{m}_2 = (\mathbf{m}_2(1), \mathbf{m}_2(2))$ is an equilibrium strategy of the defender in action space 3. First disturb $\mathbf{m}_2(1)$ by $\epsilon \in [0, \mathbf{m}_{max}(1) - \mathbf{m}_2(1))$ and adjust $\mathbf{m}_2(2)$ accordingly and we denote the new strategy as $\mathbf{m}'_2 = (\mathbf{m}'_2(1), \mathbf{m}'_2(2))$, where $\mathbf{m}'_2(1) = \mathbf{m}_2(1) + \epsilon$ and $\mathbf{m}'_2(2) = \mathbf{m}_2(2) - \epsilon$. Since $\mathbf{m}_2$ is an equilibrium strategy, the defender's utility obtained by playing $\mathbf{m}_2$, denoted as $U_d(\mathbf{m}_2)$, should be no less than that obtained by playing $\mathbf{m}'_2$, denoted as $U_d(\mathbf{m}'_2)$. So we have

$$\begin{aligned}
& U_d(\mathbf{m}'_2) - U_d(\mathbf{m}_2)\\
= \;& \bar{p}(1|\mathbf{h_1})[(\mathbf{m}_2(1)+\epsilon)(R^d(1)-P^d(1))]\\
& +\bar{p}(2|\mathbf{h_1})[(\mathbf{m}_2(2)-\epsilon)(R^d(2)-P^d(2))]\\
& -\{\bar{p}(1|\mathbf{h_1})[\mathbf{m}_2(1)(R^d(1)-P^d(1))]\\
& +\bar{p}(2|\mathbf{h_1})[\mathbf{m}_2(2)(R^d(2)-P^d(2))]\}\\
= \;& \bar{p}(1|\mathbf{h_1})[\epsilon(R^d(1)-P^d(1))]\\
& +\bar{p}(2|\mathbf{h_1})[-\epsilon(R^d(2)-P^d(2))]\\
\leq \;& 0\\
\Rightarrow \;& \bar{p}(1|\mathbf{h_1})[R^d(1)-P^d(1)] \leq \bar{p}(2|\mathbf{h_1})[R^d(2)-P^d(2)]
\end{aligned}$$

Similarly, disturb $\mathbf{m}_2(1)$ by $\epsilon \in [0, \mathbf{m_2}(1) - \mathbf{m}_{min}(1))$ in the opposite direction and adjust $\mathbf{m}_2(2)$ accordingly and we denote the new strategy as $\mathbf{m}'_2 = (\mathbf{m}'_2(1), \mathbf{m}'_2(2))$, where $\mathbf{m}'_2(1) = \mathbf{m}_2(1) - \epsilon$ and $\mathbf{m}'_2(2) = \mathbf{m}_2(2) + \epsilon$. Then we have $\bar{p}(1|\mathbf{h_1})[R^d(1)-P^d(1)] \geq \bar{p}(2|\mathbf{h_1})[R^d(2)-P^d(2)]$. Thus, we have the following equation

$$\bar{p}(1|\mathbf{h_1})[R^d(1)-P^d(1)] = \bar{p}(2|\mathbf{h_1})[R^d(2)-P^d(2)]$$

Scale both sides by dividing $\bar{p}(1|\mathbf{h_1}) + \bar{p}(2|\mathbf{h_1})$ to replace the unscaled belief with the true belief. Then we have

$$p(1|\mathbf{h_1})[R^d(1)-P^d(1)] = p(2|\mathbf{h_1})[R^d(2)-P^d(2)]$$

Since $p(1|\mathbf{h_1}) + p(2|\mathbf{h_1}) = 1$, we have

$$p(1|\mathbf{h_1}) = \frac{R^d(2) - P^d(2)}{R^d(1) - P^d(1) + R^d(2) - P^d(2)}$$

$$p(2|\mathbf{h_1}) = \frac{R^d(1) - P^d(1)}{R^d(1) - P^d(1) + R^d(2) - P^d(2)}$$

which are only related to the reward and penalty of the defender on different targets and unrelated to the specific form of $\mathbf{m}_2$.

If these two equalities hold, any such disturbance on $\mathbf{m}_2$ would not decrease the defender's utility. Therefore, any strategy in this strategy space leads to the same utility for the defender so any strategy is a potential equilibrium strategy. In addition, it can be easily proved that both $\mathbf{m}_{min}$ and $\mathbf{m}_{max}$ results in the same utility as other strategies in this space. Intuitively, we can always disturb one strategy and make the new strategy converge to either $\mathbf{m}_{min}$ or $\mathbf{m}_{max}$ without change of utility. So we assume in this case the defender chooses either $\mathbf{m}_{min}$ or $\mathbf{m}_{max}$ as his strategy.

If these two equalities do not hold together, it means either we have $\bar{p}(1|\mathbf{h_1})[R^d(1)-P^d(1)] > \bar{p}(2|\mathbf{h_1})[R^d(2)-P^d(2)]$ or have $\bar{p}(1|\mathbf{h_1})[R^d(1)-P^d(1)] < \bar{p}(2|\mathbf{h_1})[R^d(2)-P^d(2)]$. So we can always disturb one strategy in certain direction to get a better utility for the defender. Thus, the strategy will converge to either $\mathbf{m}_{min}$ or $\mathbf{m}_{max}$ whichever gives higher utility.

- **Action space 4**
  Similar to action space 3, if $p(1|\mathbf{h_1}) = \frac{R^d(1)-P^d(1)}{R^d(1)-P^d(1)+R^d(2)-P^d(2)}$ and $p(2|\mathbf{h_1}) = \frac{R^d(2)-P^d(2)}{R^d(1)-P^d(1)+R^d(2)-P^d(2)}$, any strategy in this strategy space leads to the same utility for the defender. Otherwise, the strategy will converge to either $\mathbf{m}_{min}$ or $\mathbf{m}_{max}$.

In summary, for two targets and two types of attackers, given that both types of attackers play pure behavioral strategies, either $\mathbf{m}_{min}$ or $\mathbf{m}_{max}$ leads to the highest utility for the defender at the second time step. In fact, for any one shot game, by replacing the posterior belief with the prior belief, we obtain the same result. □

Lemma 2. *At the second time step, when the defender commits to either $\mathbf{m}_{min}$ or $\mathbf{m}_{max}$, one of the two types of attackers will be indifferent among the two targets.*

Proof. Without loss of generality, we assume $\mathbf{m}_{max} = (M(1), 1-M(1))$ and $\mathbf{m}_{min} = (M(2), 1-M(2))$. If the the defender commits to $\mathbf{m}_{max}$, then we have

$$\begin{aligned} U_1^1 &= M(1)\left(P^1(1) - R^1(1)\right) + R^1(1) \\ &= \frac{R^1(1)R^1(2) - P^1(1)P^1(2)}{R^1(1) - P^1(2) + R^1(2) - P^1(1)} \\ U_1^2 &= [1 - M(1)]\left(P^1(2) - R^1(2)\right) + R^1(2) \\ &= \frac{R^1(1)R^1(2) - P^1(1)P^1(2)}{R^1(1) - P^1(2) + R^1(2) - P^1(1)} \end{aligned}$$

So we have $U_1^1 = U_1^2$, which means type 1 attacker is indifferent over both targets. Similarly, if the the defender commits to $\mathbf{m}_{min}$, we have $U_2^1 = U_2^2$. □

We name the attacker who has no preference over both targets "indifferent attacker".

Proof of Theorem 3. We claim that at both time steps, the defender choosing either $\mathbf{m}_{min}$ or $\mathbf{m}_{max}$ whichever gives higher myopic utility (let the higher utility be given by $\mathbf{m}^*$) and the two types of attackers playing best responses in every time step is a PBNE. Notice that since the defender does not know the true type of the attacker, he has to analyze both types of attackers' behavior. According to Lemma 2, there exists an indifferent attacker at the both time steps on the equilibrium path (since all types of adversary play best response to $\mathbf{m}_{min}$ or $\mathbf{m}_{max}$). At the first time step, the indifferent attacker would behave exactly the same as another type of attackers. This imitation helps to hide its true type and results in no update of the defender's belief at the second time step, which benefits itself the most since the uncertainty of his type for the defender remains unchanged. At the second time step, the indifferent attacker could choose whatever target to attack because the game ends after its attack. At probability zero histories (deviation from $\mathbf{m}^*$ by the defender in first time step or deviation by adversary in first time step), we fix the defender's belief at the second time step to be the prior.

Now we prove the optimality of this equilibrium. We want to show that no one individually wants to deviate from its equilibrium strategy at any time step.

At the first time step, the defender needs to consider his myopic utility and the future expected utility simultaneously. Since there is no belief update for any outcome after first time step, the future expected utility of the defender is a constant and is unrelated to the defender's strategy at the first time step. So the defender at the first time step only needs to choose the strategy that gives the highest myopic utility. Since $\mathbf{m}^*$ is his myopic best response, he would not deviate from this strategy.

For the two types of attackers at the first time step on the equilibrium path, if the true attacker is the indifferent attacker, any action would be its best response and it would just mimic the other attacker type's behavior to hide his type. So it would not deviate to preserve the largest amount of uncertainty of his type. If the true attacker is not the indifferent attacker, then whichever target it attacks would not update the defender's belief due to the imitation by the indifferent attacker. So it will just play the myopic best response and would not deviate from that.

For the two types of attackers at the first time step off the equilibrium path (zero probability history), the defender's belief is unaffected by its actions, hence it will play the myopic best response.

At the second time step, since the defender's belief will not be updated, the reasoning process is exactly the same as that for the first time step. The only difference is at the second time step the indifferent attacker does not need to imitate and he can attack any target. To sum up, no one wants to deviate from its equilibrium strategy alone for two-time-step game.

Now we extend two-time-step scenario to multiple time steps. For multiple time steps, at the last time step T, the case is same as the case of the second time step in two-time-step setting. From time $t = 1$ to $T - 1$, the case is same as the case of the first time step in two-time-step setting since the indifferent attacker exists at each time step and would always imitate to hide his type. The set of strategies in this equilibrium for any subgame is still an equilibrium in that the strategies of the defender and the attacker at each time step from $t = 1$ to $T-1$ are the same. Since belief is updated based on Bayes rule and players are sequentially rational, the equilibrium we claim is a PBNE.

Therefore, we conclude that for two targets and two types of attackers, there exists a PBNE in which both types of attackers would always play their myopic best responses. □